POWER OPERATIONS

ABSTRACT. Notes from the spring 2014 "Thursday seminar" at Harvard. TEXed by Akhil Mathew, who takes full responsibility for all errors.

1. 2/13: Talk by Jacob Lurie

This is motivation for what we're talking about in the Thursday seminar this semester. I'm going to start by talking about Dijkgraaf-Witten theory.

Let G be a finite group, and fix $\eta \in H^n(BG; \mathbb{C}^\times)$. If M^n is an n-dimensional manifold, you can associate to it a number $Z(M^n)$. Namely,

$$Z(M^n) = \sum_{\text{iso classes of G bundles } P} \frac{\int_{M^n} P^*\eta}{|\mathrm{Aut}(P)|}$$

The numerator is as follows. G-bundles on M are given by maps $M \to BG$. Given a G-bundle P, identified with a map $P : M \to BG$, you get a class by pulling back a class in $H^n(M; \mathbb{C}^\times)$. You can then integrate this class and then come up with a class in $\mathbb{C}^\times$.

Example 1. *If $\eta = 1$, then this is counting the number of G-bundles on M with multiplicity.*

In general, it is some twisted version of that count. This gives an invariant of n-manifolds, and this gives the top-level of a topological quantum field theory. If M is a manifold of dimension $n-1$, then there is an analogous procedure that you can perform that you can use to perform a complex *vector space.* Again, you're going to think about all G-bundles on M, but instead of thinking of them one at a time, consider the classifying space of them. That is, consider $\mathrm{Map}(M^{n-1}, BG)$. If I take this, there is a map

$$\epsilon : \mathrm{Map}(M, BG) \times M \to BG,$$

given by evaluation. Moreover, we have a projection map

$$p : \mathrm{Map}(M, BG) \times M \to \mathrm{Map}(M, BG).$$

If I start with $\eta \in H^n(BG; \mathbb{C}^\times)$, pull back along the evaluation map to give a class $\epsilon^*\eta$ in $H^n(\mathrm{Map}(M, BG) \times M; \mathbb{C}^\times)$. Then, consider the pushforward along p $\int \epsilon^*\eta \in H^1(\mathrm{Map}(M, BG); \mathbb{C}^\times)$.

Now H^1 with coefficients in $\mathbb{C}^\times$ parametrizes local systems of 1-dimensional complex vector spaces. So this class $\int \epsilon^*\eta$ classifies a local system $\mathcal{L}_M$ on $\mathrm{Map}(M, BG)$. Earlier, we got a locally constant function — we got something in H^0. We were able to sum it over the various connected components. So, we define

$$Z(M^{n-1}) = \Gamma(\mathrm{Map}(M, BG), \mathcal{L}_M).$$

This is an invariant of any $n-1$-dimensional manifold; it's a finite-dimensional complex vector space. The mapping space $\mathrm{Map}(M, BG)$ has finitely many connected components and each connected component is the classifying space of a finite group (the automorphism group of a finite G-bundle). So giving a one-dimensional local system means giving a bunch of characters of a bunch of finite groups and taking global sections means taking invariants and adding them up.

How are these $(n-1)$-dimensional invariants related to these n-dimensional invariants? To think about this, consider an $(n-1)$-manifold M^{n-1}, another N^{n-1}, and an n-manifold B giving a bordism between the two. In this case, you can look at $\mathrm{Map}(B, BG)$, and that projects onto $\mathrm{Map}(M, BG) \times \mathrm{Map}(N, BG)$. So we get maps

$$f : \mathrm{Map}(B, BG) \to \mathrm{Map}(M, BG), \quad g : \mathrm{Map}(N, BG).$$

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We get local systems $\mathcal{L}_M, \mathcal{L}_N$ on the two targets and we can pull back to get $f^*\mathcal{L}_M, g^*\mathcal{L}_N$ on $\mathrm{Map}(B, BG)$.

These are related. These local systems came from pulling back certain cohomology classes on BG and integrating them. Inside this manifold B, the fundamental cycles of M and N are homologous – B itself is a boundary between them. So in B, integrating along M and integrating along N are identified. So we should get $f^*\mathcal{L}_M \simeq g^*\mathcal{L}_N$. If you do this process at the level of cycles, you get a *preferred* choice of isomorphism $f^*\mathcal{L}_M \simeq g^*\mathcal{L}_N$.

So, given a section of $\mathcal{L}_M$, I can pull it back and push it forward to get a section of $\mathcal{L}_N$. So, I get a map

$$Z(M^{n-1}) \overset{Z(B)}{\to} Z(N^{n-1}),$$

and when M, N are empty, then the two vector spaces are canonically $\mathbb{C}$, so the map is multiplication by the number we encountered earlier.

More precisely:

Definition 1. $Z(M^{n-1}) = H^0(\mathrm{Map}(M, BG; \mathcal{L}_M)$. *(There is no higher cohomology since we are over* $\mathbb{C}$.*)*

Now there is a good way of pulling back cohomology classes, but not pushing forward. So, crucial to this discussion was the fact that homology and cohomology were the same — so I could push forward either. At the level of vector spaces, this comes from the fact that if V is a vector space over $\mathbb{C}$ with an action of a finite group G, then $V^G \simeq V_G$ canonically.

Two years ago in this seminar, we proved that there is a similar phenomenon of homology and cohomology giving you the same answer in a different context. Let X be a space which has finitely many finite homotopy groups. A space like BG is such a space. Moreover, $\mathrm{Map}(M, BG)$ is such a space. Let $\mathcal{L}$ be a local system on X of $K(n)$-local spectra. Then we can talk about $C_*(X; \mathcal{L})$ and $C^*(X; \mathcal{L})$ ("chains" and "cochains" of $\mathcal{L}$ on X).

Definition 2. *$C^*(X; \mathcal{L})$ is the homotopy limit over X of $\mathcal{L}$. ($\mathcal{L}$ is a diagram of spectra parametrized by X.) Similarly, $C_*(X; \mathcal{L})$ is the homotopy colimit of this diagram. The homotopy colimit is taken in the world of $K(n)$-local spectra.*

Two years ago, we discussed the fact that if X is as above, there is a canonical equivalence

$$C_*(X; \mathcal{L}) \simeq C^*(X; \mathcal{L}).$$

The slogan is that such a space, from the point of view of $K(n)$-local homotopy theory, behaves like a zero-dimensional space: homology and cohomology are canonically the same. A consequence is that you can run the same construction for Dijkgraaf-Witten theory.

Let's restart the discussion. Let E be a Morava E-theory of height n. I'll review what these are a little later. This is some kind of cohomology theory, and a theorem of Goerss-Hopkins-Miller implies that it is a well-structured cohomology theory. E has an essentially unique structure of an E_∞-ring. There is a multiplication in E-cohomology, but somehow it satisfies all the commutativity that you would want up to coherent homotopy.

The upshot is that it makes sense to consider $GL_1(E) \subset \Omega^\infty E$, and this is the subspace of those connected components in $\Omega^\infty E$ which are units at π_0. That's a space that you can assign to any kind of multiplication on a spectrum. You just need a way of picking out the units in π_0. But if you've got one of these fancy multiplications, then this $GL_1(E)$ is actually an infinite loop space. Any operad that acts on E as a spectrum will act on its units as a space. So $GL_1(E)$ gets an action of an E_∞-operad, so you can deloop it as many times as you want. Therefore, you can write $GL_1(E) = \Omega^\infty \mathfrak{gl}_1(E)$ where $\mathfrak{gl}_1(E)$ is a spectrum.

Now it makes sense to talk about cohomology. Start with a cohomology class $\eta_n \in H^n(BG; \mathfrak{gl}_1(E))$. By definition, this is

$$H^n(BG; \mathfrak{gl}_1(E)) = [BG, \Omega^{-n} GL_1(E)].$$

Starting with η, we can mimic all the previous constructions. At various points, we need to invoke the ambidexterity theorem to avoid distinguishing between homology and cohomology. This means that we can do the following:

(1) To every closed n-manifold M, assign $Z(M)$. Before it was a complex number; now it's a number in $\pi_0 E$.
(2) Given an $n-1$-manifold, assign to it $Z(M^{n-1})$, which is going to be an E-module spectrum.

These are related to each other in the previous way – functoriality for bordisms.

All of this is motivation; it's not going to be what the seminar is going to be about. But I'm telling you that once you choose such an element η, you can construct a bunch of manifold invariants. If you thought that this was interesting and wanted to get manifold invariants, you might ask what types of η to choose.

Now we're getting to the subject matter of this seminar.

Question: What can we say about $H^*(BG; \mathfrak{gl}_1(E))$?

That's sort of like asking what this spectrum $\mathfrak{gl}_1(E)$ looks like. Maybe I don't care about everything here. I just want to map reasonably simple things into it, like the classifying spaces of finite groups.

Let's start with this question of what $\mathfrak{gl}_1(E)$ looks like. How might you answer this question? It's a spectrum, so it's got homotopy groups. What are its homotopy groups? That's pretty easy. Note that $\mathfrak{gl}_1(E)$ is the connective delooping of $GL_1(E)$, so no homotopy groups in negative degrees, by definition. The zeroth space determines the other homotopy groups, so we get

$$\pi_* \mathfrak{gl}_1(E) = \begin{cases} 0 & \text{if } * < 0 \\ (\pi_0 E)^\times & \text{if } * = 0 \\ \pi_* E & \text{if} * > 0 \end{cases}.$$

Just knowing the homotopy groups of a spectrum is not necessarily so useful if we want to know, for instance, how to map into it. Let's try and say more.

As a warm-up, replace E by complex K-theory K. Morally, this is a special case (not quite; if you p-adically complete it, then it becomes E_1). What can we say about the homotopy groups of $\mathfrak{gl}_1(K)$? We have

$$\pi_* \mathfrak{gl}_1(E) = \begin{cases} 0 & * < 0 \\ \{\pm 1\} & * = 0 \\ \mathbb{Z} & * > 0 \text{ and even} \\ 0 & * > 0 \text{ and odd} \end{cases}.$$

Now we know something about K-theory – it has to do with vector bundles. A point in K-theory is like a vector bundle. So a point in $\mathfrak{gl}_1(K)$ should be like a vector bundle invertible with respect to the tensor product. What is true is that if you have a vector bundle invertible with respect to the tensor product, then it gives you a class in $\mathfrak{gl}_1(K)$. Now $BU(1) = \mathbb{CP}^\infty$ has a map to $\mathbb{Z} \times BU = \Omega^\infty K$. It maps, in fact, to $\{1\} \times BU$ since every line bundle has rank one. This sits inside $\{\pm 1\} \times BU = GL_1(K)$.

Moreover, this map is an infinite loop map, because the multiplication on $BU(1)$ is precisely the tensor product of vector bundles, so it maps with the way that you multiply in the classifying space. So you get an E_∞-map

$$BU(1) \to GL_1(K),$$

and at the level of homotopy, we have $BU(1) = K(\mathbb{Z}, 2)$. This map is an isomorphism in degree two and zero in all other degrees. This is somehow picking out *exactly one* of the homotopy groups of $\mathfrak{gl}_1(K)$.

You can improve this a little bit. You can assign K-theory classes to $\mathbb{Z}/2$-graded vector bundles. If you have a $\mathbb{Z}/2$-graded vector bundle $E = E_0 \oplus E_1$, you can assign to it a K-theory class which is just the formal difference $[E_0] - [E_1]$. The reason that it's not a formality is that this is multiplicative as well. The usual tensor product on *graded* vector spaces (together with the Koszul sign rule) is compatible with the way in which you multiply K-theory units. So I could have written

$$\{\pm 1\} \times BU(1) \to GL_1(K),$$

where the former is the classifying space of $\mathbb{Z}/2$-graded line bundles.

On homotopy, I get a map $\{\pm 1\} \times K(\mathbb{Z}, 2) \to GL_1(K)$ which is an isomorphism in degrees ≤ 2 and zero everywhere else.

That's kind of nice. If you look at the space $\mathbb{Z}/2 \times BU(1)$, this is the "geometric part" $GL_1^g(K)$. The infinite loop structure on $\mathbb{Z}/2 \times BU(1)$ is *not the product* because of the funny sign rule. But we can make $GL_1^g(K)$ into an infinite loop space. Note that the composite

$$GL_1^g(K) \to GL_1(K) \to \tau_{\leq 2} GL_1(K),$$

is an equivalence, by considering homotopy groups. The conclusion is:

Proposition 1.1. *$GL_1(K)$ splits as a product $GL_1^g(K) \times \tau_{\geq 4} GL_1(K)$, and this is a splitting as infinite loop spaces.*

If you didn't like this calculation, then you might like the fact that there is a "geometric" part of $\mathfrak{gl}_1(K)$ which is not so big, and some extra stuff $\tau_{\leq 4}\mathfrak{gl}_1(K)$ which is "junk." If you're interested in calculating maps from something into $\mathfrak{gl}_1$, you can easily understand maps into the geometric piece, which is pretty close to ordinary cohomology with integer coefficients. Ordinary cohomology gives you something you can use to twist K-theory by. The other thing is something you can understand using the logarithm, but maybe is not so interesting from the point of view of producing things like Dijkgraaf-Witten theory.

We want to replicate this with Morava E-theory in place of K-theory. The homotopy of $\mathfrak{gl}_1(E)$ is a big mess, but we imagine that most of it is junk, and we want to isolate the interesting part.

Let's go back to K-theory for a second. Consider the connected component $K(\mathbb{Z}, 2)$ of the geometric part. It's an infinite loop space. But it's better than that. It doesn't just have a multiplication commutative up to homotopy and homotopy and homotopy, but you can build it as a topological abelian group: you can make it commutative on the nose. One way you might try to generalize this is as follows. Here's some infinite loop space. It's big, but can I understand maps from topological abelian groups into it.

Definition 3. *Let R be any E_∞-ring. Consider the space $GL_1(R)$, a space, and the associated spectrum $\mathfrak{gl}_1(R)$. Now I'm going to define*

$$\mathbb{G}_m(R) = \mathrm{Hom}_{\mathrm{Spectra}}(H\mathbb{Z}, \mathfrak{gl}_1(E)).$$

This is an infinite loop space: in fact, I could consider the *spectrum* of maps. I prefer to think of it as a space because while I'm going to tell you something about it, I'm never going to tell you anything about the negative homotopy groups.

Equivalently, I could have said that $\mathbb{G}_m(R)$ is the collection of maps of infinite loop spaces $\mathbb{Z} \to GL_1(R)$. If I just wanted maps from $\mathbb{Z} \to GL_1(R)$, I could understand those as $\Sigma^\infty_+ \mathbb{Z} = S[t^{\pm 1}]$ (this is some kind of Laurent polynomial ring over the sphere; this observation is suggested by the calculation of the homotopy groups of this spectrum, as a Laurent polynomial ring over the homotopy groups of spheres). If I just wanted to study maps $\mathbb{Z} \to \Omega^\infty R$ of spaces, then tautologically, that's the same as maps

$$\Sigma^\infty_+ \mathbb{Z} = S^0[t^{\pm 1}] \to R.$$

If I wanted to make the first map into an E_∞-map, then I need to make $S^0[t^{\pm 1}]$ (which is an E_∞-ring) mapping to R via E_∞-ring spectra. So

$$\mathbb{G}_m(R) = \mathrm{Hom}_{E_\infty}(S[t^{\pm 1}], R).$$

We can summarize this by saying (in quotation marks) that $\mathbb{G}_m = \mathrm{Spec} S[t^{\pm 1}]$.

Goal: Let's try to compute $\mathbb{G}_m(R)$ for various R.

This is a goal that we'll do a little bit of in this lecture, but do a lot more of in the seminar. You could set it as one of the goals for your life.

I want to continue the warm-up where I'm essentially talking about K-theory, but I really want to p-adically complete, so I'm working with Morava E-theory at height one.

Special case: $R = E$, Morava E-theory of height one.

What do I mean by that? Let κ be any perfect field of characteristic p. Then there exists an essentially unique ring spectrum such that $\pi_0 E$ is the ring of Witt vectors $W(\kappa)$ and E is even periodic and the formal group associated to E (which is even periodic), i.e., $\mathrm{Spf} \pi_0 E^{\mathbb{CP}^\infty}$ is identified with the formal multiplicative group.

There is a canonical example, where you take $\kappa = \mathbb{Z}/p$ and this Morava E-theory is p-adically completed K-theory. Another example, though, which you might want to keep in mind, is where $\kappa = \overline{\mathbb{F}_p}$. In that case, we get an associated Morava E-theory which is the "maximal unramified extension" of p-adic K-theory. I'm going to write this as K_{p^∞}.

It's a corollary of the Goerss-Hopkins-Miller that this construction is totally functorial in κ so K_{p^∞} is acted on by the Galois group. In particular, it has a Frobenius automorphism. The Frobenius on $\overline{\mathbb{F}_p}$ induces a Frobenius on K_{p^∞} and the homotopy fixed points of the Frobenius gives K (p-adically).

Here are some examples. We want to compute

$$\mathrm{Hom}_{E_\infty}(S^0[t^{\pm 1}], E).$$

Let's try and do this.

The notation is potentially a little bit confusing. We're adopting two familiar notations from algebraic geometry. You can call the units of a ring R either $GL_1(R)$ or $\mathbb{G}_m(R)$. Those usually have the same meaning. I've just assigned them two different meanings. For instance, you can see pretty easily that $GL_1(R)$ is corepresentable as well. For example

$$\Omega^\infty R = \mathrm{Hom}_{E_\infty}(S^0\left\{t\right\}, R),$$

where $S^0\left\{t\right\}$ is the free E_∞-algebra on one generator. The thing that you have to beware of is that the homotopy groups of that are much more complicated than a simple polynomial ring. This guy as a spectrum looks like a sum $\bigvee_{n\geq 0} S^0_{h\Sigma_n} = \bigvee_{n\geq 0}(B\Sigma_n)_+$. If you're doing ordinary algebra, if you take the coinvariants of a group acting trivially, you get nothing, but in homotopy theory you get something very complicated. This is **not** simply a direct sum of copies of the sphere spectrum indexed by $\mathbb{Z}_{\geq 0}$, which I'll write by $S^0[t]$. So $GL_1(R)$ is given by maps from $S^0\left\{t\right\}[t^{-1}]$ into R.

Now there is a map

$$S^0\left\{t^{\pm 1}\right\} \to S^0[t^{\pm 1}],$$

and there is a map $\mathbb{G}_m(E) \to GL_1(E)$ which is sort of by definition. The $\mathbb{G}_m(E)$ are those units of E which are "strictly commutative" units in E.

So, we know of course how to compute $GL_1(E)$ at the level of homotopy groups. You might try to figure out to what extent this is off from $\mathbb{G}_m(E)$.

In the case I'm considering here, the $K(1)$-localizations of $S^0\left\{t^{\pm 1}\right\}$ and $S[t^{\pm 1}]$ different, but are not so different from one another. I'm going to just make everything an E-algebra, so the point is that $E\left\{t^{\pm 1}\right\}$ and $E[t^{\pm 1}]$. We would be done if the map

$$E\left\{t^{\pm 1}\right\} \to E[^{\pm 1}]$$

is an equivalence, but it's not. (Let's work $K(1)$-locally everywhere.)

Question: What does $L_{K(1)}E\left\{t^{\pm 1}\right\}$ look like?

Let me ask a more basic question first.

Question: What does $L_{K(1)}E\left\{t\right\}$ look like?

This is a $K(1)$-local E-algebra and it is free on one generator. What do I get if I take $\pi_0 L_{K(1)}E\left\{t\right\}$? This is some ring, and it has a simple interpretation. The elements in here are *operations* that act on π_0 of any $K(1)$-local E-algebra. If you have any $K(1)$-local E-algebra R and you have some element $x \in \pi_0 R$, then you get a map

$$\phi : L_{K(1)}E\left\{t\right\} \to R,$$

sending $t \to x$. Anything else in π_0 of the free thing goes to something else in $\pi_0 R$. So, if I have an element $c \in \pi_0 L_{K(1)}E\left\{t\right\}$, then c gives an operation $O_c : \pi_0 R \to \pi_0 R$. Namely, you make the map above ϕ, and see where $\phi(c)$ goes. This is a completely general categorical idea. It's the same reason that understanding cohomology operations classically amounts to computing the cohomology of Eilenberg-MacLane spaces.

Now, there are some operations on the K-theory of spaces that might be familiar to you.

Digression (Adams operations). If $[e] \in K^0(X)$ is a class, then according to the splitting principle, you're allowed to pretend that e splits as a direct sum of line bundles $e = L_1 \oplus \cdots \oplus L_n$. Then you can take the kth powers of those line bundles and get a new class $L_1^k \oplus \cdots \oplus L_n^k$. That determines a class $\psi^k[e]$, and this defines operations

$$\psi^k : K^0(X) \to K^0(X).$$

This is a map that you can define for any space X.

Fix a prime number p. If you take $\psi^p([\epsilon])$, then $\psi^p(\epsilon])$ looks kind of like $[\epsilon]^p$. If there was one summand, then these would be the same. In general, of course, $[\epsilon]^p$ has lots of other terms if ϵ is a sum of line bundles — but those summands are permuted by the cyclic group of order p, and the orbits have two possible shapes. It's possible to conclude that

$$\psi^p(\epsilon) \equiv \epsilon^p \mod p.$$

We can directly argue this when ϵ is a sum of line bundles, but we can actually turn this observation into another observation.

One can introduce a new operation

$$\theta_p : K^0(X) \to K^0(X),$$

for any space X, and it has the property that

$$\psi^p(x) = x^p + p\theta_p(x).$$

I'm explaining why you believe that there is such an operation — a priori there might be p-torsion so that θ_p is not uniquely defined. Or you could use the splitting principle and write down a formula directly.

This is a story you can tell for K-theory. It turns out that E is any $K(1)$-local E_∞-ring (fixed at some prime p), then one has analogous operations ψ^p, θ_p.

Theorem 1.2 (McClure)**.** *If E is any $K(1)$-local E_∞-ring spectrum (at a prime p), one has analogous operations ψ^p, θ_p.*

In ordinary K-theory, you have lots of these operations ψ^k. In p-adic K-theory, most of these operations depend on the fact that K-theory had to do with vector bundles, but ψ^p is intrinsic to complex K-theory and has nothing to do with vector bundles. These operations go

$$\theta_p, \psi^p : \pi_0 E \to \pi_0 E,$$

and they satisfy the same congruences.

(1) ψ^p is a ring homomorphism.
(2) θ_p is not a ring homomorphism: it's whatever it has to be to make the equation $\psi^p = x^p + p\theta_p(x)$.

Theorem 1.3. *There are no other operations.*

What do I mean by this? Operations that act on the cohomology of E-algebras are in bijection with $\pi_0 E\{t\}$ and the claim is that the two elements I've written down explain everything.

Theorem 1.4. $\pi_0 L_{K(1)} E\{t\} \simeq \pi_0 E[t, \theta_p t, \theta_p(\theta_p(t)), \ldots]$.

Now there is a map

$$L_{K(1)}E\{t\} \to L_{K(1)}E[t],$$

which sends t to t in homotopy. The point is that the θ_p's vanish on t in the target. So in fact, one finds (since pushouts are given by relative tensor products):

Corollary 1.5. *There is a pushout square in $K(1)$-local E-algebras*

$$\begin{array}{ccc} L_{K(1)}E\{y\} & \xrightarrow{y \mapsto \theta(t)} & L_{K(1)}E\{t\} \\ \downarrow{\scriptstyle y \mapsto 0} & & \downarrow \\ L_{K(1)}E & \longrightarrow & L_{K(1)}E[t] \end{array}.$$

In other words, $E[t]$ is the free $K(1)$-local E_∞-ring on a generator t with $\theta(t) = 0$. There is thus a cofiber sequence

$$L_{K(1)}E\{t\} \xrightarrow{\theta} L_{K(1)}E\{t\} \to L_{K(1)}E[t].$$

So if R is a $K(1)$-local E-algebra, then

$$\mathrm{Hom}_{E_\infty}(L_{K(1)}E[t], R) \to \Omega^\infty R \xrightarrow{\theta} \Omega^\infty R$$

is a fiber sequence.

So, if I wanted to compute $\mathbb{G}_m(R)$, it's just a subspace of $\mathrm{Hom}_{E_\infty}(L_{K(1)}E[t], R)$ given by those maps which send t to a unit. So we get a fiber sequence

$$\mathbb{G}_m(R) \to GL_1(R) \xrightarrow{\theta} \Omega^\infty R.$$

Conclusion: we get a long exact sequence in homotopy groups

$$\pi_n\mathbb{G}_m(R) \to \pi_n GL_1(R) \xrightarrow{\theta} \pi_n R \to \dots.$$

Consider the case where $R = E$, so the homotopy groups of R are concentrated in even degrees. Using evennness, we get short exact sequences

$$0 \to \pi_{2m}\mathbb{G}_m(R) \to \pi_{2m}(GL_1(E)) \xrightarrow{\theta} \pi_{2m}E \to \pi_{2m-1}\mathbb{G}_m(R) \to 0.$$

When $m = 0$, we should be a little bit careful. When $m = 0$, we get

$$\pi_0\mathbb{G}_m(E) = \mathrm{fib}\pi_0 GL_1(E) \to \pi_0 E.$$

Let's calculate π_0 first. Remember, $\pi_0 E = W(\kappa)$. We have this Adams operation ψ and we have this operation $\psi : W(\kappa) \to W(\kappa)$ which is actually the Frobenius (i.e., induced by the Frobenius on κ). There are no choices. This follows from the fact that $\psi^p(x) \equiv x^p \mod p$.

So what are we looking for? We want to find *units* in the Witt vectors of κ which have the property that $\theta(x) = 0$. Now $\theta(x) = \frac{\psi^p(x) - x^p}{p}$, so to say that $\theta(x) = 0$ is to say that $\psi^p(x) = x^p$. What elements are there in this ring that satisfy this equality?

Exercise: Solutions are just in bijection with the Teichmuller lifts of elements of $\kappa^\times$. For every element of κ, there's a canonical way to lift it to an element of $W(\kappa)$.

It follows that

$$\boxed{\pi_0\mathbb{G}_m(E) = \kappa^\times.}$$

Now let's compute θ in positive degrees. It's going to be a bit tricky. The reason it's easy to make a mistake because θ comes to you on π_0 and now I'm thinking of it as an operation on π_n. I can do this by reducing to the case of π_0 by considering $E^{S^{2m}}$ where

$$\pi_0 E^{S^{2m}} \simeq \pi_0 E \oplus \epsilon\pi_{2m}E,$$

where $\epsilon^2 = 0$. That's the ring structure on the function spectrum.

So I want to compute $\theta(1 + x\epsilon)$ where $x \in \pi_{2m}E$, but you're thinking of this as belonging to π_0 by multiplying by β^{-n}. By definition, since everything is torsion-free, I can write this in terms of ψ. So

$$\frac{\psi(1 + \epsilon x) - (1 + \epsilon x)^p}{p} = \frac{?? - (1 + px\epsilon)}{p}.$$

To evaluate the ??, you need to know how ψ acts on something in degree $2m$: it acts by multiplying by p^m. So in the end, you get the expression

$$p^{m-1}(\psi(x) - x))\epsilon.$$

So we need to understand the map $p^{m-1}\psi(x) - x$ as a map $W(\kappa) \to W(\kappa)$. Again, ψ is the Frobenius. There are two cases:

(1) If $m \geq 1$, then this map is congruent to the identity mod p and thus an isomorphism. Thus

$$\boxed{\pi_*\mathbb{G}_m(E) = 0 \quad * > 2}.$$

(2) We have an exact sequence

$$0 \to \pi_2\mathbb{G}_m(E) \to W(\kappa) \overset{\psi-1}{\to} W(\kappa) \to \pi_1\mathbb{G}_m(E) \to 0.$$

So, similarly, we get

$$\boxed{\pi_2\mathbb{G}_m(E) = \mathbb{Z}_p},$$

since the fixed points for the Frobenius on $W(\kappa)$ is $W(\mathbb{F}_p)$. Finally, we get

$$\boxed{\pi_1\mathbb{G}_m(E) = \operatorname{coker}(\psi - 1 : W(\kappa) \to W(\kappa)).}$$

If $\kappa = \mathbb{F}_p$), then the map we're taking the cokernel of is zero and $\pi_1\mathbb{G}_m(E) = \mathbb{Z}_p$. If we're taking κ algebraically closed, then the map $\psi - 1 \bmod p$ is $x^p - x$ and so this is a surjection. So if κ is algebraically closed, we find that $\pi_1\mathbb{G}_m(E) = 0$.

So for instance

$$\pi_*\mathbb{G}_m(K) = \begin{cases} \mathbb{F}_p^\times & * = 0 \\ \mathbb{Z}_p & * = 1,2 \\ 0 & \text{otherwise} \end{cases}.$$

However,

$$\pi_*\mathbb{G}_m(K_{p^\infty}) = \begin{cases} \overline{\mathbb{F}_p}^\times & * = 0 \\ \mathbb{Z}_p & * = 2 \\ 0 & \text{otherwise} \end{cases}.$$

I like the second calculation better because more things are zero, and because you can get the first calculation from the second by taking the fixed points of the Frobenius.

This is picking out for us inside the space of units of Morava E-theory a collection (of strictly commutative pieces) that behaves well.

Theorem 1.6 (Hopkins, Lurie)**.** *Let E be any Morava E-theory with an algebraically closed residue field κ. Then*

$$\pi_*\mathbb{G}_m(E) = \begin{cases} \kappa^\times & * = 0 \\ \mathbb{Z}_p & * = n+1 \\ 0 & \textit{otherwise} \end{cases}.$$

This is much simpler than the homotopy groups of $\mathfrak{gl}_1(E)$. If n is even the $\mathbb{Z}_p$ is happening in a place where $\mathfrak{gl}_1(E)$ is zero.

Being able to make maps into the space of units lets you make things like Dijkgraaf-Witten theory. Here's another motivation. Let me define μ_p, the version of the pth roots of unity in a ring spectrum R. By definition, this is going to be maps of E_∞-spaces $\mathbb{Z}/p\mathbb{Z} \to GL_1(R)$. So $\mu_p(R)$ is the fiber of multiplication by p on $\mathbb{G}_m(R)$. So if you know a theorem like the above, you immediately know what the pth roots of unity look like in Morava E-theory.

So we get

$$\mu_p(E) = K(\mathbb{Z}/p, n),$$

in this case. This is related to an observation, as follows. $\mu_p(R)$ is representable by a group scheme. It's maps from $E[\mathbb{Z}/p]$ to R (the "group algebra"). This ring spectrum has another incarnation. It arises from a duality phenomenon that we discussed two years ago. It is the function spectrum $E^{K(\mathbb{Z}/p,n)}$.

So

$$K(\mathbb{Z}/p, n) \simeq \mu_p(E) = \operatorname{Hom}_{E_\infty}(E^{K(\mathbb{Z}/p,n)}, E),$$

and the composite map which is a homotopy equivalence is the tautological "evaluation" map.

Corollary 1.7. *If X be any space such that π_*X is a finite p-group for all values of $*$ and 0 for $* > n$. (I.e., a finite Postnikov system where the homotopy groups stop above n.) Then the canonical map*

$$X \to \operatorname{Hom}_{E_\infty}(E^X, E),$$

is an equivalence.

This is a version of rational homotopy theory for a very restricted class of spaces.

2. 2/20 Mike Hopkins

2.1. **Introduction.** Let me remind you a little bit about where we are. So last time Jacob discussed the following situation. Let R be an E_∞-ring. Today I'm going to focus on the case where $R = K$). That means that R is a ring up to homotopy, so that $R^0(X)$ for X a space is a ring and has a group of units $R^0(X)^\times$. That's given by

$$R^0(X)^\times = [X, GL_1(R)],$$

where $GL_1(R)$ is a space sitting inside $\Omega^\infty R$. This has a natural *infinite loop space* structure

$$GL_1(R) = \Omega^\infty \mathfrak{gl}_1(R),$$

which is a *spectrum of units* in R. I.e., $GL_1(R)$ has a coherently homotopy commutative multiplication.

Last time, Jacob also introduced $\mathbb{G}_m(R)$, which is the maximal "abelian group" mapping to $GL_1(R)$ (which is only an abelian group up to coherent homotopy). So we can write

$$\mathbb{G}_m(R) = \hom_{\mathrm{Sp}}(H\mathbb{Z}, \mathfrak{gl}_1(R)),$$

or infinite loop space maps

$$\mathbb{G}_m(R) = \mathrm{Hom}_{\mathrm{InfLoop}}(\mathbb{Z}, GL_1(R)).$$

Our goal is to describe a theorem describing the structure of this when R is Morava E-theory.

We might try to calculate this directly, but $\mathfrak{gl}_1(R)$ is somewhat mysterious as a spectrum. For example, $\mathfrak{gl}_1(K) \simeq E \times B$ as a spectrum, where E is a 2-stage Postnikov system (the part of K-theory that comes from plus or minus a line) and B has the property that for all p, $\widehat{B}_p \simeq \widehat{bsu_p}$. B has the homotopy type of $\widehat{bsu_p}$ at all p-adic completions, but not integrally.

Question: How do we approach the space or spectrum of maps $H\mathbb{Z} \to \mathfrak{gl}_1(R)$?

We can use the fact that spectrum maps $H\mathbb{Z} \to \mathfrak{gl}_1(R)$ are the same thing as E_∞-maps

$$\Sigma^\infty_+ \mathbb{Z} \to R,$$

i.e.

$$\mathrm{Hom}_{\mathrm{Sp}}(H\mathbb{Z}, \mathfrak{gl}_1(R)) \simeq \mathrm{Hom}_{E_\infty}(\Sigma^\infty_+ \mathbb{Z}, R),$$

and you should think of this as analogous to the universal property of the *group ring* of a group. This has the advantage that we don't have to deal with this mysterious $\mathfrak{gl}_1$, but we do have to compute mapping spaces between E_∞-rings.

2.2. **Resolutions.** How do we compute mapping spaces between E_∞-rings such as these? The only approach is to find a *resolution* of $\Sigma^\infty_+ \mathbb{Z}$. Say, a simplicial resolution $P_\bullet \to \Sigma^\infty_+ \mathbb{Z}$ where each P_n is a *free* E_∞-ring spectrum.

I need a temporary name for the free E_∞-ring spectrum. First of all, what is a free E_∞-ring spectrum free on? It's free on a spectrum, and it's analogous to the symmetric algebra of an abelian group. Let's use that notation.

Definition 4. *Let X be a spectrum. Then $\mathrm{Sym}_*(X)$ is the wedge $\bigvee_{n \geq 0} (X^{\otimes n})_{h\Sigma_n}$, i.e. the sum of the homotopy-theoretic symmetric powers of X. This is the free E_∞-ring on X.*

Once I have a resolution $P_\bullet \to \Sigma^\infty_+ \mathbb{Z}$, then we get a *spectral sequence*

$$\pi_* \mathrm{Hom}_{E_\infty}(P_n, R) \implies \pi_*(\Sigma^\infty_+ \mathbb{Z}, R).$$

Since each $P_n = \mathrm{Sym}_*(V_n)$ for V_n a spectrum (we're assuming freeness), then

$$\mathrm{Hom}_{E_\infty}(\mathrm{Sym}_*(V_n), R) \simeq \mathrm{Hom}_{\mathrm{Sp}}(V_n, R),$$

and that's something we can calculate in terms of R-cohomology. So, I can do two things to compute mapping spaces:

- Study the R-cohomology of E_∞-rings and try to understand the E_2-term of this spectral sequence.
- We could just look around and try to find "in nature" workable explicit resolution for $P_\bullet$ with which we can compute.

Both of these are viable options, and they give something interesting. I'll talk about these both in the case of K-theory.

I want to start with Atiyah's old paper "Power operations in K-theory." When we finally get through it, it won't give the right information to feed in here, but it informs this picture quite a bit. Take the free E_∞-ring on S^0, which I've been calling $\mathrm{Sym}_*(S^0) = \bigvee_{n\geq 0} B\Sigma_{n+}$. To understand this, we might want to understand:

Question: What is the K-theory of $B\Sigma_n$?

Atiyah, when he introduced the theory of power operations in K-theory, knew that

$$K^0(\Sigma_n) \simeq \mathrm{Rep}(\Sigma_n)\widehat{\ },$$

was that the completion of the representation ring of Σ_n at the augmentation ideal. We might try to work with all this before completion.

2.3. **Representations of Σ_n.** Let me start by reminding you of the relation between $\mathrm{Rep}(\Sigma_n)$ and the theory of *symmetric functions.* Here's this basic approach, which goes back to Weyl's construction of the representations of the general linear group and to Schur's thesis. There's a sophisticated version of this in Schur's thesis in terms of polynomial functors.

Let V be an n-dimensional vector space over $\mathbb{C}$, so $V \simeq \mathbb{C}^n$. Choosing such an isomorphism, we get an action of GL_n on V. Inside GL_n, I have the diagonal matrix $(t_1, \ldots, t_n)$ where the t_i are algebraically independent. Weyl's idea is that you take $V^{\otimes k}$ which has a *commuting* action of GL_n and Σ_k. So it's a representation of the product of the groups. That means that I can decompose it into irreducible representations. We can write this

$$V^{\otimes k} \simeq \bigoplus_\pi \pi \boxtimes V_\pi,$$

where π is running through the irreducible representations of Σ_k. V_π is a representation of GL_n, and Weyl's theorem is that you can get all of them in this way (*plus duals*).

Example 2. *If π is the trivial representation, then V_π is $\mathrm{Sym}^k(V)$.*

Example 3. *If π is the sign representation, then V_π is the exterior algebra $\bigwedge^k V$.*

The other ones in between give you lots more representations of GL_n.

This gives me some element

$$\Delta \in \mathrm{Rep}(\Sigma_k) \otimes \mathrm{Rep}(GL_n).$$

Now I want to try to understand $\mathrm{Rep}(GL_n)$ via their characters, and I have this special diagonal element $(t_1, \ldots, t_n)$, and I can take the trace of this operator on any GL_n-representation. This gives a map

$$\mathrm{Rep}(\Sigma_k) \otimes \mathrm{Rep}(GL_n) \to \mathrm{Rep}(\Sigma_k) \otimes \mathrm{Sym}(t_1, \ldots, t_n),$$

taking the trace everywhere. It's not hard to see that the polynomials you get from applying the diagonal element are symmetric.

I'm not going to distinguish this element Δ from its image in $\mathrm{Rep}(\Sigma_k) \otimes \mathrm{Sym}(t_1, \ldots, t_n)$. Note that Δ is *homogeneous* of degree k. That's easy to see from tensoring all these together.

There is a useful formula for this element Δ. Fix an *equivalence relation* α on $\{1, 2, \ldots, k\}$. Inside Σ_k is the subgroup $\Sigma_\alpha \subset \Sigma_k$ consisting of permutations which *preserve* the equivalence classes (i.e., takes each equivalence class to itself). That's a product of copies of the symmetric groups on the equivalence classes. Associated to every such α relation is a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ which I'm going to write in increasing order. The *conjugacy class* of this subgroup depends only on the partition.

Let ρ_λ be the permutation subgroup corresponding to this: that is,

$$\rho_\lambda = \mathrm{Ind}_{S_\alpha}^{S_k} \mathbf{1}.$$

(Here α is an equivalence relation that yields λ; the choice doesn't matter.)

Another way to say this is that you're considering the vector space with basis on equivalence relations inducing the partition λ.

So, associated to the partition λ, I have a representation $\rho_\lambda \in \mathrm{Rep}(\Sigma_k)$. I also have a symmetric function $m_\lambda(t_1, \ldots, t_k)$, which is the sum of all monomials in the t's of type λ. If $\lambda = (2, 2, 3)$, then a typical monomial of that type is $t_i^2 t_j^2 t_k^3$. I literally sum all these monomials together.

Theorem 2.1.

$$\Delta = \sum_\lambda \rho_\lambda \otimes m_\lambda,$$

where the sum is over all partitions λ of k.

This is easy if you think about it carefully – I'll leave the proof to you.

Now, $\mathrm{Rep}(\Sigma_k)$ is a free abelian group, so I can rewrite Δ as a map

$$\mathrm{Rep}(\Sigma_k)^\vee \to \mathrm{Sym}(t_1, \ldots, t_n), \tag{1}$$

and the k means that the image is in symmetric polynomials of degree k.

The theorem that has a surprisingly easy proof is the following:

Theorem 2.2. (1) *is an isomorphism (onto the degree k symmetric polynomials) for $n \geq k$.*

Let's notice that the following diagram commutes

$$\begin{array}{ccc} & & \mathrm{Sym}(t_1, \ldots, t_{n+1}) \\ & \nearrow & \downarrow \\ \mathrm{Rep}(\Sigma_k)^\vee & \longrightarrow & \mathrm{Sym}(t_1, \ldots, t_n) \end{array}$$

and for n large, the degree k part of $\mathrm{Sym}(t_1, \ldots, t_{n+1})$ stabilizes in n.

Proof. First note that both sides of (1) are free of the same rank. The representations of the symmetric group Σ_k is free of the rank which is the number of conjugacy classes in Σ_k, i.e., the number of partitions of k. For the symmetric functions side, there are a lot of ways one might prove this, but maybe one reason is that the algebra of symmetric functions is the polynomial algebra on the elementary symmetric functions. So, suffices to show that we get an epimorphism.

For that, you could use the formula directly. I'd have to find a homomorphism out of the representations of the symmetric group that took ρ_λ to 1 and the other $\rho_{\lambda'}$ to zero. For instance, I could look carefully at the ρ_λ by ordering them.

There is another approach that gives you a little more information. Instead of looking at the homogeneous part of degree k, we can look at the whole algebra of symmetric functions. Let

$$R_* = \bigoplus \mathrm{Rep}(\Sigma_k)^\vee,$$

and that's a *graded ring*. In fact, we have a map

$$\mathrm{Rep}(\Sigma_k) \otimes \mathrm{Rep}(\Sigma_l) \simeq \mathrm{Rep}(\Sigma_k \times \Sigma_l) \to \mathrm{Rep}(\Sigma_{k+l}),$$

and dualizing gives the multiplication maps we want.

The map

$$R_* \to \mathrm{Sym}(t_1, \ldots, ...),$$

is a *ring homomorphism.* You can prove that directly from the properties of that element Δ, but it's more conceptual than that and is more conceptual. If you want to learn more about these ideas, you should read about polynomial functors. Anyway, this winds up being a consequence of the fact that the trace of a tensor product of matrices is a product of traces.

My goal is to prove that the map is surjective, so let's write down some elements in R_*. Let $\tau_k \in \mathrm{Rep}(\Sigma_k)^\vee$ be the delta function at the trivial representation. So

$$\tau_k(W) = \dim \mathrm{Hom}_{\Sigma_k}(\mathbb{C}, W), \quad W \in \mathrm{Rep}(\Sigma_k).$$

There are these elements τ_k in the dual representation ring, and where do these go under the map? That's easy to check from the examples. Now τ_k goes to the trace of $(t_1, \ldots, t_k)$ on symmetric $\mathrm{Sym}^k V$ which is the monomial symmetric function. (The coefficient of z^k in $\frac{1}{(1-t_1 z)\ldots(1-t_n z)}$.) Those coefficients form a system of polynomial generators for the ring of symmetric functions. So we get that the map

$$R_* \to \mathrm{Sym}(t_1, t_2, \dots)$$

is a surjection, hence a ring isomorphism. □

We also get this very useful fact, that:

Corollary 2.3. *R_* is a polynomial ring on these elements $\tau_1, \tau_2, \dots$.*

This has a very nice consequence. The representation ring of Σ_k has a canonical basis, from the irreducible representations. So it comes to you with a self-duality $\mathrm{Rep}(\Sigma_k) \simeq \mathrm{Rep}(\Sigma_k)^\vee$. So you also learn something about the representations of the symmetric group from this. For instance, τ_k corresponds to the trivial representation. And these sums of all

$$\bigoplus_{k \geq 0} \mathrm{Rep}(\Sigma_k)$$

forms a ring (isomorphic to R_* under this self-duality!), where given a representation of Σ_k and Σ_l, you $\boxtimes$ them together and induce via $\Sigma_k \times \Sigma_l \to \Sigma_{k+l}$. The trivial representations τ_k form a system of polynomial generators for the ring. So these permutation representations we looked at earlier form a basis for the representation ring from the symmetric group.

2.4. K**-theory again.** This isn't quite the thing we wanted. We were looking at $\widehat{K}_p(B\Sigma_n)$ and we wanted to know the homotopy groups, which I'll write as

$$K_{p*}(X) = \pi_* \varprojlim_{r \to \infty} K/p^r \wedge X.$$

So then we have

$$K_{p*}(B\Sigma_n) \simeq \mathrm{Hom}(K_p^0(B\Sigma_n), \mathbb{Z}_p),$$

because these things are free, and this is somehow related to representations of Σ_n by the Atiyah-Segal completion theorem.

We can work that out, and if you do, you get the answer Jacob advertised, which is the p-adic K-theory of the free E_∞-ring on S^0. This was first done by Hodgkin in 1972 which was the original source for this. The title was "The K-theory of some well-known spaces – I, QS^0." There isn't a paper number II. In this paper, he goes through shows what you have to do to modify to put in the completion. The other place where these results are collected is in McClure's paper in the "H_∞-book."

There's some structure that exists that he doesn't use, though. These are the *power operations.* Let X be a space, and consider a cohomology class,

$$X \to K(\mathbb{Z}/2, m),$$

and we raise that to the kth power, and that extends over

$$X^k \to (X^k)_{h\Sigma_k} \to K(\mathbb{Z}/2, mk),$$

and this can be done by naturality when $X = K(\mathbb{Z}/2, m)$ and a connectivity argument gives you a unique way of doing this. Then you restrict to the diagonal and get a cohomology class of $X \times B\Sigma_k = (X)_{h\Sigma_k}$, so given an element in the *homology* of $B\Sigma_k$ we can pair against it and get a new element in the cohomology of X. For each time we choose an element in the homology of $B\Sigma_k$, we get a cohomology operation.

Atiyah was doing all this to talk about this in the language of K-theory. We start with a representation V of $GL_n(\mathbb{C})$ (the defining one) which gives a K-theory class

$$BGL_n(\mathbb{C}) \to K,$$

and we raised it to the kth power and restrict to the diagonal to get an element in $K^0(B\Sigma_k \times BGL_n(\mathbb{C}))$ with the property that if I restrict to $\{*\} \in B\Sigma_k$, then I get $V^{\otimes k}$. This gives me the same element Δ I was dealing with earlier. Now $K^0(BGL_n(\mathbb{C}))$ is a formal power series rather than a polynomial ring, but it's much the same.

Atiyah did this a little differently, and I think there's an interesting story there that nobody has ever worked out. Instead of completing the representation ring, you can work with the representation ring to actually work with *equivariant K-theory.* It seems like there's a way of looking at symmetric functions and so forth as attached to the multiplicative group, and there's probably a pretty analog of these theories if there's a good equivariant version.

This leads to the following fact. **Elements of $K_{p*}(B\Sigma_k)$ give power operations in K-theory** (i.e., in the K-theory of spaces).

There are a couple of facts that I forgot to mention:

(1) $K_p^0(BG)$ has rank equal to the number of conjugacy classes of elements of p-power order.
(2) $K_p^0(B\Sigma_k)$ has a basis over the p-adics given by the ρ_λ where each $\lambda = (\lambda_1, \dots, \lambda_i)$ is a partition into powers of p.

That has an interesting consequence, since these are all permutation representations. In the category of spectra, I have an inclusion map

$$B\Sigma_{\lambda+} \to B\Sigma_{k+},$$

and a transfer map in the reverse direction. The classes in $B\Sigma_{k+}$ come from taking the trivial map $B\Sigma_{\lambda+} \to S^0 \to K$ and then using the transfer to get a map $B\Sigma_{k+} \to S^0 \to K$. It follows from this that the $B\Sigma_{k+}$ are wedges of spheres $K(1)$-locally by using a bunch of these transfer maps to split. At higher chromatic levels, the $B\Sigma_{k+}$ are generally not wedges of spheres.

The calculation that Jacob described was that if you take $K_{p_*}(\mathrm{Sym}(S^0))$ is the free θ-algebra

$$\mathbb{Z}_p[x, \theta x, \theta(\theta(x)), \dots],$$

where you p-adically complete everywhere. You can work out from Hodgkin's calculation that you get a polynomial algebra, but you want to understand the action of θ. Let me say a brief word about that. But first, let me remind you what θ is.

Consider $K_{p*}(B\Sigma_p)$. There are only two partition representations to look at here. There's $(1, 1, \dots, 1)$ or (p). These are permutation representations, so these give you (by transfer maps) $B\Sigma_{p+} \to S^0$. I'm going to call one of them ϵ (the trivial one) and the other one I'm going to take to be something a little different, the transfer map $S^0 \simeq B\Sigma_{p-1+} \to B\Sigma_{p+}$ corresponding to the p-dimensional permutation representation. Then ϵ, Tr give an equivalence $K(1)$-locally

$$B\Sigma_{p+} \simeq S^0 \vee S^0.$$

Now I define an element $\theta \in K_{p*}(B\Sigma_{p+})$ via

$$\theta(\epsilon) = 0, \quad \theta(V_p) = -1, \quad \psi(1) = 1, \;\psi(V_p) = 0.$$

A clean way to think about power operations is to consider an arbitrary p-adically complete E_∞-algebra E over p-adic K-theory. For example, K-valued cochains on a space. Given an element in $\pi_0 E$, given by a map $x : S^0 \to E$ of spectra, I apply the extended power construction to get

$$B\Sigma_{p+} \to (E^{\otimes p})_{h\Sigma_p} \to E,$$

and the way that works out, that's given by $\pi_0 E \otimes K_{p,*}(B\Sigma_p)$. So I can apply either ψ or θ to it. That will define new elements in $\pi_0 E$.

Upshot: we define natural transformations

$$\psi, \theta : \pi_0 E \to \pi_0 E,$$

where E is any p-adically complete E_∞-algebra under K.

Let me state some general facts. There is a trivial element in $K_{p*}(B\Sigma_{p+})$ coming from inclusion of a point, which gives an operation $e : \pi_0 E \to \pi_0 E$. This operation is the one which pretends that Σ_p isn't there, so that

$$e(x) = x^p, \quad x \in \pi_0 E.$$

We get this relation

$$\psi(x) = x^p + p\theta(x), \quad x \in \pi_0 E \tag{2}$$

and you can check that ψ is a ring homomorphism. So $\theta : \pi_0 E \to \pi_0 E$ is some natural transformation. On the free E_∞-algebra, you have a canonical class $x \in \pi_0$, and the theorem is that you get a polynomial algebra on the iterates of θ applied to x.

Example 4. *Let X be a space (or spectrum) such that $K_{p*}(X)$ is free of finite rank and concentrated in even degrees. Then let's look at the AHSS*

$$H_*(B\Sigma_p, K_{p*}(X^{\otimes p})) \implies K_*((X^{\otimes p})_{h\Sigma_p}).$$

By the Künneth formula $K_{p*}(X^{\otimes p}) \simeq K_{p*}(X)^{\otimes p}$. As a Σ_p-representation, this is a sum of trivial Σ_p-modules and things which are free under restriction to the p-Sylow. This spectral sequence has to collapse because of evenness. What you learn from this calculation is that

$$K_*(X^{\otimes p}_{h\Sigma_p}) \simeq \mathrm{Sym}^p(K_{p*}(X)) \oplus \theta(K_{p*}X).$$

That implies that everything in the K-theory of Free(S^0) comes from the generators and θ's of that. Then you can count rank. This is an argument that does generalize, and you can describe this in terms of Morava E-theory.

2.5. **Symmetric powers.** Jacob gave a *resolution* of this Laurent series E_∞-ring Σ^∞_+ as a $K(1)$-local E_∞-ring. There happens to be another resolution that is just setting there in nature that has to do with partition complexes which gives you the same spectral sequence. Let me just mention it to you know. You could set this up without understanding so much of the structure of power operations.

There is a different type of symmetric powers $SP^n(S^0)$, the *nth symmetric power of the sphere.* The nth symmetric power of a space X is $SP^n(X) = (X^n)_{\Sigma_n}$ — the literal, not homotopy, quotient. This is a continuous functor, so we get $\Sigma SP^n(X), SP^n(\Sigma X)$. Therefore, we can apply this to spectra. We can make a spectrum whose mth space is $SP^n(S^m)$. These sit in a filtration

$$SP^1(S^0) \to SP^2(S^0) \to \dots$$

of spectra, and the colimit, by the Dold-Thom theorem, is $H\mathbb{Z}$. The first one is the sphere spectrum. This gives a way of mediating between S^0 and $H\mathbb{Z}$.

Let's apply to everything $\Sigma^\infty\Omega^\infty$. We get

$$\Sigma^\infty_+ QS^0 \to \dots \to \Sigma^\infty_+\mathbb{Z},$$

and we've written $\Sigma^\infty_+\mathbb{Z}$ as a colimit of various Σ^∞_+ things. So we get an inverse system converging to $\mathrm{Hom}_{E_\infty}(\Sigma^\infty_+\mathbb{Z}, \cdot)$.

Example 5. *$SP^2(S^0)/S^0 \simeq \Sigma\mathbb{RP}^\infty$.*

There is a fibration sequence

$$\mathbb{RP}^\infty \to S^0 \to SP^2(S^0),$$

and you can apply $\Sigma^\infty_+\Omega^\infty$ to this, to get pushout squares

$$\begin{array}{ccc} \Sigma^\infty_+\Omega^\infty\Sigma^\infty\mathbb{RP}^\infty & \longrightarrow & S^0 \\ \downarrow & & \downarrow \\ \Sigma^\infty_+ QS^0\Sigma^\infty_+ & \longrightarrow & \Omega^\infty SP^2(S^0) \end{array}$$

Now you can show that

$$\Sigma^\infty_+\Omega^\infty \mathrm{SP}^2(S^0) \to \Sigma^\infty_+ H\mathbb{Z},$$

is an equivalence in p-adic K-theory. **This generalizes as well.**

There is a theorem:

Theorem 2.4 (Arone-Dywer)**.**

$$SP^nS^0/SP^{n-1}S^0 \simeq \Sigma^\infty(S^n \wedge P_n)_{h\Sigma_n},$$

where P_n is some sort of partition complex.

At a prime p, the $SP^n(S^0)$ only change at pth powers, and if you're interested in the nth Morava E-theory, you only go up to $SP^{p^n}(S^0)$.

3. 2/27: Nat Stapleton

Let E be Morava E-theory E_n (i.e., height n) for the rest of the talk. In particular, $E_n^0 = W(k)[[v_1, \dots, v_{n-1}]]$. This is an E_∞-ring spectrum. Therefore, for a space X, there are power operations

$$P_m : E^0(X) \to E^0(E\Sigma_m \times_{\Sigma_m} X^m) \to E^0(B\Sigma_m \times X) \simeq E^0(B\Sigma_m) \otimes_{E^0} E^0(X).$$

We get operations $E^0(X) \to E^0(X)$ from this by considering the dual of $E^0(B\Sigma_m)$ (which is finitely generated and *free* over E^0). For each element in $(E^0(B\Sigma_m))^\vee \simeq \hat{E_0}(B\Sigma_m)$, we get a choice of power operation

$$E^0(X) \to E^0(X).$$

If you unwrap what P_m is doing, you find that it's multiplicative but *not* additive. In general, there is a *formula* for what P_m does to a sum. Let $Tr_{i,j} : E^0(B\Sigma_i \times B\Sigma_j) \to E^0(B\Sigma_m)$ be the transfer map along the inclusion $\Sigma_i \times \Sigma_j \to \Sigma_m$ (where $i + j = m$). Here's the basic fact.

Proposition 3.1.

$$P_m(x + y) = P_m(x) + P_m(y) + \sum_{0<i<m} Tr_{i,m-i}(P_i(x) \otimes P_{m-i}(y)).$$

Thus, if we quotient by an ideal $I_{\text{tr}} \subset E^0(B\Sigma_m)$ generated by the images of these transfer maps, the resulting composite

$$E^0(X) \xrightarrow{P_m} E^0(B\Sigma_m) \otimes E^0(X) \to E^0(X) \otimes E^0(B\Sigma_m)/I_{\text{tr}}$$

is a *ring homomorphism.*

Now, dualizing, $(E^0(B\Sigma_m)/I_{\text{tr}})^\vee \subset \hat{E_0}(B\Sigma_m)$ is a submodule, and it corresponds to those operations which are *additive.* This is equivalent to the degree m primitives in the Hopf algebra $\hat{E_0}(\text{Sym}^*(S^0))$ which are primitive, with respect to the comultiplication induced by transfer maps.

Let's think about equivariant K-theory for a moment. In this case, we have a map

$$P_m : K(X) \to R(\Sigma_m) \otimes K(X),$$

for a space X, and if we quotient by a certain ideal $I \subset R(\Sigma_m)$, we get a ring map.It turns out that $R(\Sigma_m)/I \simeq \mathbb{Z}$. Why is that? We can think in terms of conjugacy classes. Say I have a conjugacy class of maps $\mathbb{Z} \to \Sigma_m$, which corresponds to an isomorphism class of $\mathbb{Z}$-sets of size m, we're asking if the map $\mathbb{Z} \to \Sigma_m$ factors through $\Sigma_i \times \Sigma_j$ for some $i + j = m$. Only the *transitive* $\mathbb{Z}$-sets with m-elements end up surviving. But there's only one transitive $\mathbb{Z}$-set with m elements. (Anyway, this quotient $R(\Sigma_m)/I$ was calculated last time??)

Anyway, we get a ring map

$$K(X) \to (R\Sigma_m/I) \otimes K(X) \simeq K(X),$$

and we get precisely the mth Adams operation. This motivates our goal.

Goal: We want to understand $E^0(B\Sigma_m)/I_{\text{tr}}$ as a ring.

Our tool for going about this is going to be versions of character theory. Let's recall what's going on in the classical case in character theory. Let G be a finite group. The character map

$$R(G) \to \text{Cl}(G, \mathbb{C}),$$

where the latter is the ring of class functions on G with values in the complex numbers. If I form $\mathbb{C} \otimes R(G)$, then I get an isomorphism

$$\mathbb{C} \otimes R(G) \simeq \text{Cl}(G, \mathbb{C}).$$

We don't need all of $\mathbb{C}$ either, we only need $\mathbb{Q}^{\text{ab}}$, the maximal cyclotomic extension.

Now $\mathbb{Q}^{\text{ab}}$ has an interesting property. If you consider $\mathbb{G}_m$ and consider the torsion points, then if you pull this back to $\mathbb{Q}^{\text{ab}}$, that's the minimal ring over which the divisible group associated to $\mathbb{G}_m$ is isomorphic to the constant group scheme $\mathbb{Q}/\mathbb{Z}$.

Example 6. *$R(G) = K_G(*)$ and we are considering the character map $R(G) \to \mathrm{Cl}(G; \mathbb{Q}^{\mathrm{ab}})$ where the latter is the product $\prod_{G^{\mathrm{conj}}} \mathbb{Q}^{\mathrm{ab}}$ and this is the $\mathbb{Q}^{\mathrm{ab}}$-rational cohomology*

$$(H\mathbb{Q}^{\mathrm{ab}})^0(LBG),$$

of the free loop space LBG. The components of the free loop space LBG come from the conjugacy classes of G.

So we get this map

$$K_G(*) \to (H\mathbb{Q}^{\mathrm{ab}})(LBG),$$

and it's going from a height one thing (K-theory) to a height zero thing (rational cohomology) and when you tensor up you get an isomorphism.

Questions:

(1) Can this generalize to higher heights?
(2) Can this make sense as a map of cohomology theories?

The answer to both questions is **yes,** and that's what we're about to do.

For every number $0 < t < n$, we're going to try and construct a map from E (nth Morava E-theory) and lands in some height t cohomology theory. Recall that E has a formal group, the universal deformation $\mathbb{G}_E$ which we're going to think of as a p-divisible group. In other words, we are going to think of it in terms of its prime power torsion, and that's actually equivalent to $\mathbb{G}_E$. Recall also that

$$E^0(B\mathbb{Z}/p^k) \simeq E^0[[x]]/([p^k](x)),$$

and this is the ring of functions on the p^k-torsion of $\mathbb{G}_E$.

Now let's take $K(t)$ be height t Morava K-theory. We don't need to know what this is, but we do need to understand localization with respect to this. I need to understand what $L_{K(t)}E$. What are the resulting coefficients? In this case,

$$\pi_* L_{K(t)}E \simeq \pi\widehat{(E)[u_t^{-1}]}_{(p,u_1,\ldots,u_{t-1})}.$$

Here $\pi_0 E \simeq W(k)[[u_1, \ldots, u_{n-1}]]$. If you want to understand this algebro-geometrically, look at Nat's paper with Aaron and Eric to give an interpretation of this ring.

Now let X be a finite G-CW complex. Consider the function spectrum into E from the Borel construction X_{hG}. The slogan is that you can describe this in terms of $L_{K(t)}E$ and the $(n-t)$th free loop space of X_{hG}:

$$L_{K(t)}E^{X_{hG}} \text{``}\sim\text{''} (L_{K(t)}E)^{\mathcal{L}^{n-t}X_{hG}}.$$

But two things go wrong:

(1) Different loops.
(2) Different coefficients.

Definition 5. *Let $G_p^h = \mathrm{Hom}(\mathbb{Z}_p^h, G)$ which means commuting h-tuples of p-torsion elements in X.*

Let X be a G-CW complex. Consider $X_{hG} = X//G$ and we want $E^0(X//G)$. There is a nice functor from finite G-CW complexes to finite G-CW complexes that was invented by Hopkins and Kuhn. Let X be a finite G-CW complex. We get a new one given by $\coprod_{\alpha \in G_p^n} X^{\mathrm{im}\alpha}$.

Now here's a fact

$$\mathcal{L}^h(X//G) \overset{\mathrm{def}}{=} (\coprod_{\alpha \in G_p^n} X^{\mathrm{im}\alpha})_{hG} \simeq \mathrm{Hom}(*//\mathbb{Z}_p^h, X//G).$$

When $X = *$, then

$$\mathcal{L}^h(BG) = \mathcal{L}^h(*//G) \simeq \mathrm{Hom}(B\mathbb{Z}_p^h, BG) \simeq \coprod_{\alpha \in G_p^h/\sim} BC(\mathrm{im}\alpha),$$

where $\mathcal{L}^h BG$ really means the p-adic free loop space.

One last observation. If I have a cohomology theory, then I can get a new one by precomposing with $\mathcal{L}^h(??)$.

Next I want to say coefficients. We construct a faithfully flat $(L_{K(t)}E_n)_0$-algebra C_t such that we have an isomorphism

$$C_t \otimes_E \mathbb{G}_E \simeq \mathbb{G}_{L_{K(t)}E} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^t.$$

This is an isomorphism of p-divisible groups. There's a really big ring that makes this work.

The character map. If X is a finite G-CW complex, I want a map

$$E^0(X//G) \to C_t \otimes_{(L_{K(t)}E_n)_0} (L_{K(t)}E_n)^0(\mathcal{L}^n(X//G)).$$

Theorem 3.2 (Stapleton)**.** *If we start with $C_t \otimes E^0(X//G)$, then the above map becomes an isomorphism.*

When $t = 0$, this is Hopkins-Kuhn-Ravenel. When $n = 1$, I believe that this is due to Adams. When $X = *$, we get an isomorphism

$$E^0(BG) \otimes C_t \simeq \prod_{\alpha \in G_p^{n-t}/\sim} C_t^0(B\mathrm{Cent}(\alpha)).$$

When $t = 0$, we get the rational Morava E-theory of BG after tensoring up.

(**I was unable to finish TEX ing this talk.**)

4. Jacob 3/27

Let κ be a perfect field, $\mathbb{G}_0$ a height n formal group over κ. There is an even periodic ring spectrum E associated to this data, so that $\pi_0 E$ is the Lubin-Tate ring: $W(\kappa)[[v_1, \dots, v_{n-1}]]$. Throughout this lecture, I'm going to call this R. I should really say the formal spectrum of R, but I'll be a bit sloppy and consider Spec$R$ and say that it classifies formal deformations of $\mathbb{G}_0$. In particular, there is a deformation of $\mathbb{G}_0$ living over Spec$R$. This formal group is the formal spectrum of $E^0(\mathbb{CP}^\infty)$.

Now, previously we talked about $GL_1(E)$ and $\mathbb{G}_m(E)$, which is the *space* of infinite loop maps $\mathbb{Z} \to GL_1(E)$. Our goal is to compute $\mathbb{G}_m(E)$. Here is the result.

Theorem 4.1. *If κ is algebraically closed, then*

$$\pi_i(\mathbb{G}_m(E)) = \begin{cases} \kappa^\times & i = 0 \\ \mathbb{Z}_p & i = n+1 \\ 0 & \textit{otherwise} \end{cases}.$$

Now, for any E, $\mathbb{G}_m(E)$ can be described as the space of maps of E_∞-algebras over E from $E \wedge \Sigma^\infty_+ \mathbb{Z} \to E$. Now I'm going to write

$$E[t^{\pm 1}] \simeq E \wedge \Sigma^\infty_+ \mathbb{Z},$$

since on homotopy groups it looks like a Laurent polynomial ring. I'm going to go a bit further and $K(n)$-localize this, which does a completion with respect to the maximal ideal $\mathfrak{m}_R \subset R$, so you get a completed Laurent polynomial ring. You get things like $1 + pt + p^2t^2 + \dots$. The reason I want to say it this way is that we know a lot about the structure of $K(n)$-local E_∞-algebras over E.

I'm going to give a recap of Lukas's lecture.

Definition 6. *An E-module M is* ***free*** *if it is a sum of copies of E (no shifts allowed).*

If you look at free E-modules, then these form a homotopy theory; there are spaces between them. You could take the homotopy category of that, and that's equivalent to the category of free R-modules. A more general observation is that if M is free and you want π_0 of the space of E-module maps from $M \to N$ that is just the set of R-linear maps $R \to N$.

If you look at E_∞-algebras over E (let's stick to the $K(n)$-local world), this is a homotopy theory which has a forgetful functor to $K(n)$-local E-modules. This forgetful functor has a left adjoint, which takes an E-module and takes the free E_∞-algebra on it. It takes an E-module M to the sum $\bigoplus_{m \geq 0} (M^{\otimes m})_{h\Sigma_m}$. I should $K(n)$-localize everything everywhere. This forgetful functor is monadic.

Let T^{top} be the functor from $K(n)$-local E-modules to itself by taking the free $K(n)$-local E_∞-algebra on a given module and forgetting the algebra structure. Then this is a monad and algebras over it are $K(n)$-local E_∞-E-algebras.

Fact: If M is a finite free E-module, then if I take $L_{K(n)}M^{\otimes d}_{h\Sigma_d}$ then this is also a finite free E-module.

In this case, the construction $M \mapsto L_{K(n)}M^{\otimes d}_{h\Sigma_d}$ is a functor T_d from finite free E-modules to finite free E-modules. At the level of homotopy categories, this implies that there exists a functor T_d from finite free R-modules to finite free R-modules, such that if you take π_0 of the symmetric dth power in homotopy theory, then you get this.

Remark: When $n = 0$, this is the usual symmetric power functor, but when $n > 0$, it is a little more complicated.

You can do an algebraic version of this construction earlier, of $L_{K(n)} \bigoplus_d (M^{\otimes d})_{h\Sigma_d}$. I can try to do this algebraically. In particular,

$$M \mapsto T^{\mathrm{alg}}(M) = \bigoplus_{d\geq 0} T_d M,$$

is a functor from finite free R-modules to free (no longer finite) R-modules. I'd like to say that this is a monad, and by taking Ind-objects I can take this on the category of free R-modules to itself.

Now note that

$$T_d(M \oplus N) \simeq \bigoplus_{d_1+d_2=d} T_{d_1}(M) \otimes T_{d_2}(N).$$

So in particular

$$T(M \oplus N) \simeq T(M) \otimes T(N).$$

Long ago, there was a characterization of the category of T-algebras. I want to emphasize what it buys you: we cared initially about T^{top}-algebras (i.e., E_∞-algebras).

Definition 7. *Let M be a $K(n)$-local E-module. An* ***approximation to*** *M is a free R-module M_0 with a map $M_0 \to \pi_0 M$ such that if I identify M_0 with a lot of copies of R, then I can think of this map as giving a map $\bigoplus_I E \to M$; I want this to exhibit M as a $K(n)$-localization of $\bigoplus_I E$.*

For example if M_0 is a finite free R-module, then I'd want $M_0 \to \pi_0 M$ to be an isomorphism and M to be even. These will exist if and only if M is a sum of copies of E in the $K(n)$-local category; all the E-modules we will consider today will satisfy this condition.

Suppose I give you a $K(n)$-local E-module M and a map $M_0 \to \pi_0 M$ which is an approximation. So M_0 is a free R-module, so I can apply T^{alg} to it and get another free R-module. I can also apply T^{top} and form the free E_∞-algebra generated by M. Now the point is that there is a natural map

$$T^{\mathrm{alg}}(M) \to \pi_0 T^{\mathrm{top}}(M)$$

which is also an approximation. So T^{alg} is something we can do in algebra that reflects more or less what happens in homotopy when we form free E_∞-algebras, but it doesn't $\mathfrak{m}$-adically complete everywhere.

Suppose that A is an E_∞-algebra over E, $K(n)$-local. In thi situation, $\pi_0(A)$ is a T^{alg}-algebra. T^{alg} is an algebraic gadget designed to capture the extra structure on π_0 of an E_∞-ring. Suppose we're given an approximation $A_0 \to \pi_0 A$ where A_0 is coming to us as an R-free T^{alg}-algebra. Let's add the caveat that A_0 is free as a T^{alg}-algebra, so that A_0 comes to me as $T^{\mathrm{alg}}(M_0)$. Then, we can compute maps from A into some other $K(n)$-local E_∞-E-algebra totally algebraically, using maps of T^{alg}-algebras.

So far I've used one nontrivial fact, which is that T makes some sense: these symmetric powers end up preserving finite free modules. Now I want to use some more specific information from Lukas's lecture. Let me introduce a bit of notation. Look at the Frobenius $\mathrm{Spec}\kappa \to \mathrm{Spec}\kappa$. We started with a formal group $\mathbb{G}_0$ defined over $\mathrm{Spec}\kappa$ and we can pull it back along the Frobenius, call that $\mathbb{G}_0^{(p)}$. There is another thing which is to take $\mathbb{G}_0$ itself, and I get the geometric Frobenius

$$\mathbb{G}_0 \to \mathbb{G}_0^{(p)},$$

which is an isogeny of formal groups. I could do this with any power of the Frobenius. Let's say I did it with the dth power of the Frobenius. I could study deformations of that diagram. Now $\mathrm{Spec}\kappa$ sits inside $\mathrm{Spec}R$ over which we have the universal deformation of $\mathbb{G}_0$. I could consider data of the following type. I could consider complete local rings A that map to Spec$R$ in two different ways u, q equipped with a map $u^*\mathbb{G} \to q^*\mathbb{G}$ such that when I pull back to the residue field, I get the dth power of the Frobenius. There is a universal example Spec$A$ fitting into such a diagram. Let me call this SpecR_d, which has two maps to Spec$R$, u, q. On SpecR_d, there are two formal groups, one of which is a quotient of the other.

There are maps

$$\mathrm{Spec}R_d \times_{\mathrm{Spec}R} \mathrm{Spec}R_{d'} \to \mathrm{Spec}R_{d+d'},$$

that correspond to composing Frobenii.

Definition 8. *Let M be an R-module. Adams operations for M consist of the following data:*

(1) For each $d \geq 0$, a map

$$\psi^{(d)} : q^*(M) \to u^*(M),$$

satisfying "transitivity."

(2)

Now, we can think of this in another way. M is a rule that takes a deformation of $\mathbb{G}_0$ over Spec$A$ *to an A-module. Adams operations are rules that which take deformations of the Frobenius map over A to A-modules.*

Anyway, the category of modules with Adams operations is symmetric monoidal and we can talk about algebras here. We'll call them algebras with Adams operations.

Note: over R/p, we have the Frobenius isogeny from $\mathbb{G}$ to itself. So that is a deformation of the Frobenius. It's not the universal deformation of the Frobenius, but it is a deformation, and we get a map $R_1 \to R/p$. Note that the two maps Spec$R/p \to$ Spec$R$ are not the same; one is the obvious map, and one is the obvious map composed with the Frobenius.

Adams operations for M give us in particular a canonical map $M/p \to M/p$. Let's call this map ψ^{Frob}. It's the Adams operation corresponding to the Frobenius itself. If A is an R-algebra with AO's, then we have this map $\psi^{\mathrm{Frob}} : A/p \to A/p$. Now A/p is another map $A/p \to A/p$ which is the Frobenius itself.

Definition 9. *We say that A* ***satisfies the Wilkerson congruence*** *if these two maps I described are the same: ψ^{Frob} and the Frobenius.*

Theorem 4.2. *If M is a free R-module, then $T^{\mathrm{alg}}(M)$ is the free R-algebra with Adams operations satisfying the Wilkerson congruence.*

In order to go further, I need to invoke another nontrivial property of T^{alg}. **I need to use the fact that $T^{\mathrm{alg}}(M)$ is a polynomial ring.** As always, I'm assuming M is a free module.

For every M, I need to give a map $q^*M \to u^*M$ where q, u are the maps above. That's interchangeable with a map $M \to q_*u^*M$ which is happening over Spec$R$. What is q_*u^*M as an R-module? Now I need to tensor up with R_d and then regard that as an R-module using the other R-algebra structure on R_d. It turns out that R_d is finite flat as an R-module with respect to either of these maps. This is the same data as a map $R_d^\vee \otimes_R M \to M$. I need to give you maps like this for every d and they need to be compatible with each other. There is a map Spec$R_d \times_{\mathrm{Spec}R}$ Spec$R_{d'} \to$ Spec$R_{d+d'}$ which corresponds to a map on algebras going in the other direction, $R_{d+d'} \to R_d \otimes_R R_{d'}$. If I dualize everything, I get a map $R_d^\vee \otimes_R R_{d'}^\vee \to R_{d+d'}^\vee$. I should be careful which R-module structure I'm dualizing over.

Anyway,

$$\Gamma = \bigoplus_{d \geq 0} R_d^\vee$$

and these maps make Γ into a graded ring. In degree zero, we have R. It's not a commutative ring unless $n = 0, 1$. It's a version of the Hecke algebra for $GL_n(\mathbb{Z}_p) \subset GL_n(\mathbb{Q}_p)$. When $n = 1$,

$$\Gamma = \mathbb{Z}_p[\psi^p].$$

Definition 10. *Adams operations for M means that M is a Γ-module.*

This is a very concrete definition.

Example 7. *R is a T^{alg}-algebra. ψ^{Frob} is a map which, at least in the $n = 1$ case, is the lift of Frobenius to the Witt vectors.*

Definition 11. *An **augmented T^{alg}-algebra** is a T^{alg}-algebra A with a map $\epsilon : A \to R$.*

Given an augmented algebra $\epsilon : A \to R$, it has a kernel called the augmentation ideal $I \subset A$. We can look at A/I^2. The claim is that A/I^2 inherits the structure of a T^{alg}-algebra. This can be checked using the Wilkerson criterion if we know that I/I^2 is free.

Example 8. *Let's take $T^{\mathrm{alg}}(M)$ where M is a free R-module; this has an obvious augmentation sending $M \mapsto 0$. That has some augmentation ideal I. If I look at $T^{\mathrm{alg}}(M)/I^2$, this is universal among augmented T^{alg}-algebras receiving a map from M such the augmentation ideal squares to zero.*

Definition 12. *I'm going to write $\Delta(M) = I/I^2$ where I is the augmentation ideal of $T^{\mathrm{alg}}(M)$.*

Consider augmented T^{alg}-algebras with $I^2 = 0$. This maps to the category of R-modules via taking indecomposables Q (which in this case is the augmentation ideal); that has a left adjoint. This is a monadic adjunction and gives a monad Δ on the category of free R-modules.

What can we say about this functor Δ? What happens if we take $\Delta(M \oplus N)$? This is indecomposables in $T^{\mathrm{alg}}(M \oplus N) \simeq T^{\mathrm{alg}}(M) \otimes T^{\mathrm{alg}}(N)$ and if I take indecomposables in a tensor product, it's indecomposables in the factors. So Δ commutes with direct sums and with filtered colimits. Therefore, it's given by tensoring with some (R, R)-bimodule.

Definition 13. $\Delta = \Delta(R)$.

This really knows Δ of anything else since $\Delta(M) \simeq \Delta \otimes_R M$.

Example. If $n = 1$, $\Delta = W(\kappa)[\theta^p]$.

Now Δ is some linearized version of T^{alg}-algebras. What if I had an augmented algebra and the augmentation ideal didn't square to zero? So the conclusion is that $Q(A)$ is always a Δ-module. Another way of describing what Δ is that it consists of operations that act on the indecomposables whenever we have a T^{alg}-algebra.

If A, B are augmented T^{alg}-algebras, then if $I_B^2 = 0$, then maps of augmented algebras $A \to B$ are the same as maps of Δ-modules $Q(A) \to Q(B)$.

Goal: given $K(n)$-local E_∞-algebras (over E) A, B, we want to compute the space of maps $A \to B$. We have a particular A, B in mind. The particular A we have in mind is $L_{K(n)}Ept^{\pm 1}]$ and $B = E$ itself. We will occasionally invoke properties special to this situation. How can we compute this? Let's try to compute this by resolving A. The first special property is that A has an approximation. If I take $R[t^{\pm 1}]$, it maps to $\pi_0 A$ and is not an isomorphism, but becomes one after completion.

Let's resolve A. We can take the free E_∞-algebra generated by A, $T^{\mathrm{top}}(A)$, which maps to A, and it's free. There's a pretty standard resolution we can use here,

$$T^{\mathrm{top}} T^{\mathrm{top}}(A) \rightrightarrows T^{\mathrm{top}}(A),$$

which gives a tautological resolution of A by free T^{top}-algebras. Let me call this $T^{\mathrm{top}\bullet+1}(A)$. Then A is the geometric realization of this simplicial gadget. Therefore, if I want to compute maps $A \to B$, this can be identified with the totalization of a cosimplicial space of maps from each of these things into A. Now, whenever you have a cosimplicial space, there is a spectral sequence that you might try to use to say something about the totalization of that space. It's what people call a fringed spectral sequence. You start with information about the mapping spaces of the various pieces and hope to get the homotopy groups of tot.

$$E_1^{s,t} = \pi_s \mathrm{Map}_{E_\infty}(T^{t+1}(A), B) \simeq \pi_s \mathrm{Map}_{E-mod}(T^t(A), B),$$

and you might hope to understand this. There's an entirely parallel algebraic story you can write here. Instead of writing T^{top} to A, you can apply T^{alg} to A_0.

Now note also

$$\pi_s \mathrm{Map}_{E_\infty}(T^{t+1}(A), B) \simeq= \pi_0 \mathrm{Map}_{E_\infty}(T^{t+1}(A), B^{S^s}),$$

and we have a purely algebraic description of B^{S^s}? So basically the E_1 term of this spectral sequence as a purely algebraic description as

$$\mathrm{Hom}_{\mathrm{aug},T^{\mathrm{alg}}}(T^{\mathrm{alg}\bullet+1}A_0, R \oplus \pi_s E).$$

This is fringed, so not everything here is an abelian group — some of these things are sets. But fortunately, when s is odd, nothing happens. If s is odd, these things vanish.

When $s = 0$, we have a cosimplicial set. The equalizer that survives to E_2 is going to be the maps $A_0 \to B_0$ of T^{alg}-algebras. So, maps of algebras which are compatible with Adams operations. Algebraically, one finds that this forces $t \in R[t^{\pm 1}]$ to go to Teichmuller lifts of $\kappa^\times$.

For the rest, I have to take indecomposables levelwise and obtain a simplicial Δ-module. Now when I do this, I get R itself by arguing with André-Quillen cohomology....

We get $\mathrm{Ext}^t_\Delta(R, \pi_s E)$. But be careful. Here R is a Δ-module because it's $Q(A_0)$, not the usual one. We're going to continue this next time...Here R is a Δ-module because it's $Q(A_0)$, not the usual one.

5. Jacob 4/3

A, B are $K(n)$-local E_∞-algebras over a Morava E-theory E. We want to compute the space of maps $\mathrm{Hom}_E(A, B)$ of E-algebras, or at least compute its homotopy groups. For example, we might want to compute $\mathbb{G}_m(E)$, so that $A = E[\mathbb{Z}]$ ($K(n)$-localized) and $B = E$ itself.

In the previous lecture, we described a method of computing this: resolve A by free E-algebras, and use that to get a spectral sequence. If you understand what the free E-algebras look like, you can name the E_2-page of the spectral sequence.

Let $R = \pi_0 E$, which is isomorphic to $W(k)[[v_1, \ldots, v_{n-1}]]$. It has a universal property. We started with a formal group $\mathbb{G}_0$ of a perfect field κ. R was the ring which classifies universal deformations. For each $d \geq 0$, I introduced a ring R_d. $\mathrm{Spec} R_d$ maps to $\mathrm{Spec} R$ in two different ways. In other words, R_d is an R-algebra in two different ways. R_d classifies isogenies of formal groups $\mathbb{G} \to \mathbb{G}'$ which deform the Frobenius isogeny $\mathbb{G}_0 \to \mathbb{G}_0^{(p)}$ on $\mathrm{Spec}\kappa$.

Consider the map $\mathrm{Spec} R_d \xrightarrow{u} \mathrm{Spec} R$ which classifies $\mathbb{G}$ and $q : \mathrm{Spec} R_d \to \mathrm{Spec} R$ classifies the quotient. I'm going to think of R_d as an (R, R)-bimodule where the left action comes from u and the right action from q.

We had the notion of a T^{alg}-algebra, which was an R-algebra A equipped with "Adams operations," $q^*(A) \to u^*(A)$. I could describe this by saying that it's an algebra A over R with a map $A \to q_* u^* A$. We needed a certain congruence to be satisfied by this map, for the Frobenius isogeny. When $d = 1$, there is a canonical thing that maps to $\mathrm{Spec} R_d$, which is $\mathrm{Spec} R/p$, because there is a canonical deformation of the Frobenius: the Frobenius itself. This map $A \to q_* u^* A$ is supposed to be, after mapping to R/p, given by raising to the pth power.

I'm going to call this structure a congruence Γ-algebra rather than a T^{alg}-algebra.

We could have turned this operation around and thought about this as a map from $R_d^\vee \otimes A \to A$ and, if I add these all up, it's an action of $\Gamma = \bigoplus_{d \geq 0} R_d$ on A.

Suppose A is an E-algebra and we have a map of congruence Γ-algebras $A_0 \to \pi_0 A$ where A_0 is free as an R-module and $\pi_0 A$ is the completion of A_0, $\pi_1(A) = 0$, then we saw that we could completely understand algebraically what the homotopy groups of a resolution of A by free E_∞-algebras would look like. In this situation, we get a fringed spectral sequence

$$E_1^{s,t} = \mathrm{Hom}_{\mathrm{congruence}\Gamma-\mathrm{alg},\mathrm{augmented}}((T^{\mathrm{alg}})^{(s+1)} A_0, \pi_0 B \oplus \pi_t B) \implies \pi_{t-s} \mathrm{Hom}_E(A, B).$$

Under the additional assumption that A_0 was smooth over R, which is satisfied in our example here, we could identify the E_2-page in an explicit manner. If A_0 is smooth, then $E_2^{s,t} = \mathrm{Ext}^{s,t}_\Delta(Q(A_0), \pi_t B)$ for $t > 0$. In the case $s = t = 0$, then it's the collection of congruence Γ-algebras $A_0 \to B_0$.

Consider $A_0 = R[t^{\pm 1}]$. Why or how is this a congruence Γ-algebra? What are the Adams operations in this case? Let's remember how they were defined. For a general A, what are you supposed to do? You're supposed to map

$$\pi_0 A \to \pi_0 A \otimes_R R_d,$$

and here R_d classifies subgroups. R_d has an incarnation as a quotient of $E_n^0(B\Sigma_{p^d})$, as Nat explained. What this map given by? We're supposed to take the map

$$\pi_0 A \to A^0(B\Sigma_{p^d+}) \simeq \pi_0 A \otimes_R E^0(B\Sigma_{p^d+}),$$

and then mod out by the transfer ideal in that ring. We got a map from maps $S \to A$ to maps $B\Sigma_{p^d+} \to A$ that came from the E_∞-structure on A.

E_∞-algebras are complicated – there's an extended power construction going on here. In our case, the E_∞-structure came about because we had a group algebra structure. If we have a map $* \to \mathbb{Z}$ sending $* \to 1$, we can do the extended power construction on $\mathbb{Z}$ itself. This means that the power operations are really simple in this case. So the map $\pi_0 A \to \pi_0 A \otimes_K E^0(B\Sigma_{p^d+})$ is just $t \mapsto t^{p^d}$.

Now in our example, $A = R[t^{\pm 1}]$ and B is R itself, and we want to compute maps $A \to B$ which are compatible with Adams operations. This map is determined by where it sends t, to some $x \in R$. This element x is not arbitrary; it has to satisfy a condition indicating that it's compatible with Adams operations. Now R_d receives two maps u^*, q^* from R. The compatibility condition is that a certain diagram has to commute. It amounts to saying that for any d, $q^*(x) = u^*(x)^{p^d} \in R_d$.

Can we find elements in R that satisfy this condition? Remember that R looks like $W(\kappa)[[v_1, \ldots, v_{n-1}]]$ and it receives a unique map from $W(\kappa)$ which is unique on the residue field, by the universal property of the Witt vectors. In particular, I can map $W(\kappa)$ into R, and into R_d, and the diagram will commute if I twist by the appropriate power of the Frobenius. So if I can find solutions to the equation $q^*(x) = u^*(x)^{p^d}$ in the Witt vectors (where $q^*(x)$ is a Frobenius times $u^*(x)$) then I'm good. Now it turns out that any Teichmuller representative of something in $\kappa^\times$ is a solution. That gives us a lot of candidates for things that might come from an E_∞-map.

I claim that **there are no other solutions.** For this, it's convenient to think about another Adams operation one could make, which is the pth power $\mathbb{G} \to \mathbb{G}$, which is an isogeny that deforms the isogeny of multiplication by p. Multiplication by p on $\mathbb{G}$ over κ factors through the nth power of Frobenius plus an isomorphism. So in other words I can think of the pth power as a deformation of the Frobenius defined over Spec$R$. What are we saying? We get a map Spec$R \to$ SpecR_n such that Spec$R \to$ Spec$R_n \rightrightarrows$ Spec$R$ is the identity and then not the identity.

Let's suppose we have an element that satisfies $q^*(x) = u^*(x)^{p^d}$ for R_n, so that I can pull it back to get an element which satisfies an equation in Spec$R$. When I pull it back, it tells me that $\alpha(x) = x^{p^n}$ for some map α of R. Since the Teichmuller lifts satisfy this, I can assume $x \equiv 1 \mod \mathfrak{m}_R$, so that $x^{p^n} \equiv 1 \mod \mathfrak{m}_R^2$ while $\alpha(x) \in 1 + \mathfrak{m}_R^2$. Since α is an isomorphism, this type of argument shows that x must be 1 if it is $\equiv 1 \mod \mathfrak{m}_R$.

So we know what happens at the fringe. What we really need to understand is the rest of this spectral sequence — these Ext's over Δ. For that, I need to remind you what Δ is. Recall that Δ is a ring which acts on $Q(A)$ for any congruence Γ-algebra A. Rather, Δ is the best ring that does. We defined it last time by thinking about the following adjunction. We have augmented congruence Γ-algebras A such that the augmentation ideal squared was 0, and we said that this category was equivalent to the category of modules over some ring, and we defined Δ to be that ring. Δ is analogous to the Dyer-Lashof algebra.

What is the relationship between Γ and Δ? Any Δ-module is a Γ-module. You don't change the underlying set via this construction. This is implemented by having a map $\Gamma \to \Delta$. This map is *rationally* an isomorphism. That's because the difference Γ and Δ is the difference between Γ-algebras and congruence Γ-algebras.

What's happening integrally? Let's say M is a Γ-module which is R-free. We can ask when is M a Δ-module? (If M is a Δ-module, then it is so in one way.) Now this is true when we take the square-zero extension $R \oplus M$ which has Γ-algebra structure, and then it satisfies the congruence criterion. Let's think of this as a bunch of maps $\psi^{(p^d)} : M \to M \otimes_R R_d$ for the universal order p^d subgroup. For example, we can look at

$$M \xrightarrow{\psi^{(1)}} M \otimes_R R_1 \to M \otimes_R R/p,$$

using $R_1 \to R/p$, and this has to be **zero** because the congruence criterion implies that this should be the Frobenius. So, the answer to the question "when is M a Δ-module" is answered by saying that the above compositions have to be zero.

I want to make a trivial consequence of this relation. The collection of Γ-modules forms a tensor category. I can tensor together two modules with Adams operations. So let's restrict our attention to R-free Γ-modules and inside here we can consider the R-free Δ-modules. These form an **ideal** in the tensor category.

What are some examples of Γ-modules which are Δ-modules? First let's give a non-example. If we take the unit object of Γ-modules, so that's R itself. That corresponds to taking the **identity map** $q^*R \to u^*R$. This doesn't satisfy the congruence condition.

What are some examples? $\pi_t B$ is always a Δ-module if B is an E-algebra. For example, $\pi_2 E$ is the Lie algebra Γ-module, which is a Δ-module roughly because the derivative of the Frobenius is zero. The construction that takes $M \to M \otimes_R \omega$ takes Γ-modules to Δ-modules. **This is an equivalence of categories. So a Γ-module is a Δ-module if and only if it can be divided by ω.** We get that there is an **isomorphism** of rings

$$\Delta \xrightarrow{\beta} \omega \otimes \Gamma \otimes \omega^{-1}$$

This is certainly an isomorphism if we invert p, so we need to check that the map is surjective. (This was sketched in the talk.)

Anyway, if we want

$$\mathrm{Ext}^{s,t}_{\Delta}(Q(A), \pi_t B) = \mathrm{Ext}^{s,t}_{\Gamma}(Q(A_0) \otimes \omega^{-1}, \pi_t B \otimes \omega^{-1}).$$

Let's specialize to our example, so that $A = L_{K(n)}E[\mathbb{Z}]$ and $B = E$. Then $A_0 = R[t^{\pm 1}]$ so that $Q(A_0) = R$. This is R, but not as a Γ-module. This is called $\det = Q(A_0)$. This det is a free R-module on one generator $t-1$. We have a map of rank 1 free modules

$$R\{T-1\} \to R\{T-1\} \otimes_R R_d,$$

and we already computed earlier that $T \mapsto T^{p^d}$ so $T - 1 \mapsto (T^{p^d} - 1) \mod (T-1)^2 = p^d(T-1) \mod (T-1)^2$. So this det as an R-module is not this usual R, but where the action of $\psi^{(d)}$ is multiplied by a p^d. It's different enough such that $Q(A)$ is actually a Δ-module.

So basically we have to calculate

$$E_2^{s,t} = \mathrm{Ext}^s_{\Gamma}(\det/\omega, \omega^{t-1}),$$

and the answer is going to be that these vanish unless $s = n-1, t = n$, in which case we get $\mathbb{Z}_p$.

I'm going to get us set up for this calculation, by describing **how to compute** Ext **in the category of Γ-modules**. This is a purely algebraic story, which you can ask over any ring. You can either try to resolve the domain, or resolve the target by injectives. Or you could try to resolve N by "cofree" things.

Note: for any R-module N_0, we can make a cofree Γ-module on N_0 which is given by the following construction: $\prod_{d \geq 0} N_0 \otimes_R R_d$. There is a natural equivalence $\mathrm{Hom}_{\Gamma}(M, N) \simeq \mathrm{Hom}_R(M, N_0)$. You can also do this for Ext's.

So if I choose M such that M is R-free, then I can compute Ext groups by using a cofree resolution. Let's get started. Starting with an arbitrary Γ-module N, we can map N to $\prod_{d\geq 0} N \otimes_R R_d$. This map is injective.

Notation: for any sequence $m_1, \ldots, m_k$, we let $\mathrm{Flag}_{m_1,\ldots,m_k}$ be the scheme parametrizing sequences $\mathbb{G}^0 \to \mathbb{G}^1 \to \cdots \to \cdots \to \mathbb{G}^k$ each of which deforms the m_ith powers of the Frobenius and such that the kernel of the long map is killed by p.

Theorem 5.1. *$\mathrm{Flag}_{m_1,\ldots,m_k}$ is finite flat over* $\mathrm{Spec}R$.

Let Flag_k be the disjoint union of all these. There aren't very many. We let $\mathcal{O}_{\mathrm{flag}}$ be the ring of functions on Flag_k, which has a whole bunch of different R-algebra structures since we get $k+1$ different deformations of the original formal group. I'll just remember the first and the last.

I'll also define $\widetilde{Flag}_k^d = \mathrm{Flag}_k \times_{\mathrm{Spec}R} \mathrm{Spec}R_d$.

Fix N. Consider $\prod_{d\geq 0} N \otimes_R \mathcal{O}(\widetilde{Flag}_k^d) \otimes_R R_d$. The idea is to make k vary here, and use this to get a cosemisimplicial resolution of N.