Contents

Chapter 13
Various topics

This chapter is currently a repository for various topics that may or may not reach a status worthy of their own chapters in the future, but in any event should be included.

§1 Linear algebra over rings

1.1 The determinant trick

We want to understand what $IN = N$ means.

Let $I \subset R$ and ${}_RM$ finitely generated. Let $E = \mathrm{End}_R(M)$, which is not commutative in general. We may view M as an E-module ${}_EM$. Since every element in R commutes with all of E, E is an R-algebra (i.e. There is a homomorphism $R \to E$ sending R into the center of E).

Lemma 1.1 (Determinant Trick)

1. *Every $\phi \in E$ such that $\phi(M) \subset IM$ satisfies a monic equation of the form $\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0$, where each $a_i \in I$, i.e. ϕ is "integral over I".*

2. *$IM = M$ if and only if $(1-a)M = 0$ for some $a \in I$.*

Proof. (1) Fix a finite set of generators, $M = Rm_1 + \cdots + Rm_n$. Then we have $\phi(m_i) = \sum_j a_{ij}m_j$, with $a_{ij} \in I$ by assumption. Let $A = (a_{ij})$. Then these equations tell us that $(I\phi - A)\vec{m} = 0$. Multiplying by the adjoint of the matrix $I\phi - A$, we get that $\det(I\phi - A)m_i = 0$ for each i. It follows that $\det(I\phi - A) = 0 \in E$. But $\det(I\phi - A) = \phi^n + a_1\phi^{n-1} + \cdots + a_n$ for some $a_i \in I$.

(2) The "if" part is clear. The "only if" part follows from (1), applied to $\phi = \mathrm{id}_M$. ▲

Remark Determinant trick (part 2) actually includes Nakayama's Lemma, because if I is in $\mathrm{Rad}\, R$, $(1-a)$ is a unit, so $M = (1-a)M = 0$.

Corollary 1.2 *For a finitely generated ideal $I \subset R$, $I = I^2$ if and only if $I = eR$ for some $e = e^2$.*

Proof. $(\Leftarrow)$ clear.

$(\Rightarrow)$ Apply determinant trick (part 2) to the case $M = {}_RI$. We get $(1-e)I = 0$ for some $e \in I$, so $(1-e)a = 0$ for each $a \in I$, so $a = ea$, so I is generated by e. Letting $a = e$, we see that e is idempotent. ▲

Corollary 1.3 (Vasconcelos-Strooker Theorem) *For any finitely generated module M over* any *commutative R. If $\phi \in \mathrm{End}_R(M)$ is onto, then it is injective.*

Proof. We can view M as a module over $R[t]$, where t acts by ϕ. Apply the determinant trick (part 2) to $I = t{\cdot}R[t] \subset R[t]$. We have that $IM = M$ because ϕ is surjective, so $m = \phi(m_0) = t{\cdot}m_0 \in IM$. It follows that there is some $th(t)$ such that $(1 - th(t))M = 0$. In particular, if $m \in \ker\phi$, we have that $0 = (1 - h(t)t)m = 1 \cdot m = m$, so ϕ is injective. ▲

1.2 Determinantal ideals

Definition 1.4 An ideal $I \subset R$ is called *dense* if $rI = 0$ implies $r = 0$. This is denoted $I \subset_d R$. This is the same as saying that ${}_R I$ is a faithful module over R.

If I is a principal ideal, say Rb, then I is dense exactly when $b \in \mathcal{C}(R)$. The easiest case is when R is a domain, in which case an ideal is dense exactly when it is non-zero.

If R is an integral domain, then by working over the quotient field, one can define the rank of a matrix with entries in R. But if R is not a domain, rank becomes tricky. Let $\mathcal{D}_i(A)$ be the i-th *determinantal ideal* in R, generated by all the determinants of $i \times i$ minors of A. We define $\mathcal{D}_0(A) = R$. If $i \geq \min\{n, m\}$, define $\mathcal{D}_i(A) = (0)$.

Note that $\mathcal{D}_{i+1}(A) \supset \mathcal{D}_i(A)$ because you can expand by minors, so we have a chain

$$R = \mathcal{D}_0(A) \supset \mathcal{D}_1(A) \supset \cdots \supset (0).$$

Definition 1.5 Over a non-zero ring R, the *McCoy rank* (or just *rank*) of A to be the maximum i such that $\mathcal{D}_i(A)$ is dense in R. The rank of A is denoted $rk(A)$.

If R is an integral domain, then $rk(A)$ is just the usual rank. Note that over any ring, $rk(A) \leq \min\{n, m\}$.

If $rk(A) = 0$, then $\mathcal{D}_1(A)$ fails to be dense, so there is some non-zero element r such that $rA = 0$. That is, r zero-divides all of the entries of A.

If $A \in \mathbb{M}_{n,n}(R)$, then A has rank n (full rank) if and only if $\det A$ is a regular element.

Exercise 13.1 Let $R = \mathbb{Z}/6\mathbb{Z}$, and let $A = diag(0,2,4)$, $diag(1,2,4)$, $diag(1,2,3)$, $diag(1,5,5)$ (3×3 matrices). Compute the rank of A in each case.

Solution

A	$\mathcal{D}_1(A)$	$\mathcal{D}_2(A)$	$\mathcal{D}_3(A)$	
$diag(0,2,4)$	(2)	(2)	(0)	$3 \cdot (2) = 0$, so $rk = 0$
$diag(1,2,4)$	R	(2)	(2)	$3 \cdot (2) = 0$, so $rk = 1$
$diag(1,2,3)$	R	R	(2)	$3 \cdot (2) = 0$, so $rk = 2$
$diag(1,5,5)$	R	R	R	so $rk = 3$

1.3 Lecture 2

Let $A \in \mathbb{M}_{n,m}(R)$. If R is a field, the rank of A is the dimension of the image of $A : R^m \to R^n$, and $m - rk(A)$ is the dimension of the null space. That is, whenever $rk(A) < m$, there is a solution to the system of linear equations

$$0 = A \cdot x \tag{13.1}$$

which says that the columns $\alpha_i \in R^n$ of A satisfy the dependence $\sum x_i \alpha_i = 0$. The following theorem of McCoy generalizes this so that R can be any non-zero commutative ring.

Theorem 1.6 (McCoy) *If R is not the zero ring, the following are equivalent:*

1. *The columns $\alpha_1, \ldots, \alpha_m$ are linearly dependent.*
2. *Equation 13.1 has a nontrivial solution.*
3. *$rk(A) < m$.*

Corollary 1.7 *If $R \neq 0$, the following hold* *(a) If $n < m$ (i.e. if there are "more variables than equations"), then Equation 13.1 has a nontrivial solution.*

(b) R has the "strong rank property": $R^m \hookrightarrow R^n \Longrightarrow m \leq n$.

(c) R has the "rank property": $R^n \twoheadrightarrow R^m \Longrightarrow m \leq n$.

(d) R has the "invariant basis property": $R^m \cong R^n \Longrightarrow m = n$.

Proof (Proof of Corollary). (a) If $n < m$, then $rk(A) \leq \min\{n, m\} = n < m$, so by Theorem 1.6, Equation 13.1 has a non-trivial solution.

$(a \Rightarrow b)$ If $m > n$, then by (a), any R-linear map $R^m \to R^n$ has a kernel. Thus, $R^m \hookrightarrow R^n$ implies $m \leq n$.

$(b \Rightarrow c)$ If $R^n \twoheadrightarrow R^m$, then since R^m is free, there is a section $R^m \hookrightarrow R^n$ (which must be injective), so $m \leq n$.

$(c \Rightarrow d)$ If $R^m \cong R^n$, then we have surjections both ways, so $m \leq n \leq m$, so $m = n$. ▲

Corollary 1.8 *Let $R \neq 0$, and A some $n \times n$ matrix. Then the following are equivalent (1) $\det A \in \mathcal{C}(R)$; (2) the columns of A are linearly independent; (3) the rows of A are linearly independent.*

Proof. The columns are linearly independent if and only if Equation 13.1 has no non-trivial solutions, which occurs if and only if the rank of A is equal to n, which occurs if and only if $\det A$ is a non-zero-divisor.

The transpose argument shows that $\det A \in \mathcal{C}(R)$ if and only if the rows are independent. ▲

Proof (Proof of the Theorem). $0 = Ax = \sum \alpha_i x_i$ if and only if the α_i are dependent, so (1) and (2) are equivalent.

$(2 \Rightarrow 3)$ Let $x \in R^m$ be a non-zero solution to $A \cdot x = 0$. If $n < m$, then $rk(A) \leq n < m$ and we're done. Otherwise, let B be any $m \times m$ minor of A (so B has as many columns as A, but perhaps is missing some rows). Then $Bx = 0$; multiplying by the adjoint of B, we get $(\det B)x = 0$, so each x_i annihilates $\det B$. Since $x \neq 0$, some x_i is non-zero, and we have shown that $x_i \cdot \mathcal{D}_m(A) = 0$, so $rk(A) < m$.

$(3 \Rightarrow 2)$ Assume $r = rk(A) < m$. We may assume $r < n$ (adding a row of zeros to A if needed). Fix a nonzero element a such that $a \cdot \mathcal{D}_{r+1}(A) = 0$. If $r = 0$, then take x to be the vector with an a in each place. Otherwise, there is some $r \times r$ minor not annihilated by a. We may assume it is the upper left $r \times r$ minor. Let B be the upper left $(r+1) \times (r+1)$ minor, and let $d_1, \ldots, d_{r+1}$ be the cofactors along the $(r+1)$-th row. We claim that the column vector $x = (ad_1, \ldots, ad_{r+1}, 0, \ldots, 0)$ is a solution to Equation 13.1 (note that it is non-zero because $ad_{r+1} \neq 0$ by assumption). To check this, consider the product of x with the i-th row, $(a_{i1}, \ldots, a_{im})$. This will be equal to a times the determinant of B', the matrix B with the $(r+1)$-th row replaced by the i-th row of A. If $i \leq r$, the determinant of B' is zero because it has two repeated rows. If $i > r$, then B' is an $(r+1) \times (r+1)$ minor of A, so its determinant is annihilated by a. ▲

Corollary 1.9 *Suppose a module ${}_RM$ over a non-zero ring R is generated by $\beta_1, \ldots, \beta_n \in M$. If M contains n linearly independent vectors, $\gamma_1, \ldots, \gamma_n$, then the β_i form a free basis.*

Proof. Since the β_i generate, we have $\gamma = \beta \cdot A$ for some $n \times n$ matrix A. If $Ax = 0$ for some non-zero x, then $\gamma \cdot x = \beta A x = 0$, contradicting independence of the γ_i. By Theorem 1.6, $rk(A) = n$, so $d = \det(A)$ is a regular element.

Over $R[d^{-1}]$, there is an inverse B to A. If $\beta \cdot y = 0$ for some $y \in R^n$, then $\gamma B y = \beta y = 0$. But the γ_i remain independent over $R[d^{-1}]$ since we can clear the denominators of any linear dependence to get a dependence over R (this is where we use that $d \in \mathcal{C}(R)$), so $By = 0$. But then $y = A \cdot 0 = 0$. Therefore, the β_i are linearly independent, so they are a free basis for M. ▲

§2 Finite presentation

2.1 Compact objects in a category

Let $\mathcal{C}$ be a category. In general, colimits tell one how to map *out of* them, not into them, and there is no a priori reason to assume that if $F : I \to \mathcal{C}$ is a functor, that

$$\varinjlim_i \mathrm{Hom}(X, Fi) \to \mathrm{Hom}(X, \varinjlim Fi) \tag{13.2}$$

is an isomorphism. In practice, though, it often happens that when I is *filtered*, the above map is an isomorphism. For simplicity, we shall restrict to the case when I is a *directed* set (which is naturally a category); in this case, we call the limits **inductive.**

Definition 2.1 The object X is called **compact** if (13.2) is an isomorphism whenever I is inductive.

The following example motivates the term "compact."

Example 2.2 Let $\mathcal{C}$ be the category of Hausdorff topological spaces and *closed inclusions* (so that we do not obtain a full subcategory of the category of topological spaces), and let X be a compact space. Then X is a compact object in $\mathcal{C}$.

Indeed, suppose $\{X_i\}_{i\in I}$ is an inductive system of Hausdorff spaces and closed inclusions. Suppose given a map $f : X \to \varinjlim X_i$. Then each X_i is a closed subspace of the colimit, so we need to show that $f(X)$ lands inside one of the X_i. This will easily imply compactness.

Suppose not. Then $f(X)$ contains, for each i, a point x_i that belongs to no $X_j, j < i$. Choose a countable subset $T \subset I$ (if I is finite, then this is automatic!). For each $t \in T$, we get an element $x_t \in f(X)$ that belongs to no X_i for $i < t$. Note that if $t' \in T$, then it follows that $X_{t'} \cap \{x_t\}$ is finite.

In particular, if $F \subset \{x_t\}$ is *any* subset, then $X_{t'} \cap F$ is closed for each $t' \in T$. Thus $\varinjlim_T X_{t'}$ contains the set F as a closed subset, and since this embeds as a closed subset of $\varinjlim X_i$, F is thus closed in there too. The induced topology on $\{x_t\}$ is thus the discrete one.

We have thus seen that the set $\{x_t\}$ is an infinite, discrete closed subset of $\varinjlim X_i$. However, it is a subset of $f(X)$ as well, which is compact, so it is itself compact; this is a contradiction.

This example allows one to run the "small object argument" of Quillen for the category of topological spaces, and in particular to construct the *Quillen model structure* on it. See [Hov07]. As an simple example, we may note that if we have a sequence of closed subspaces (such as the skeleton filtration of a CW complex)

$$X_1 \subset X_2 \subset \dots$$

it then follows easily from this that (where $[K, -]$ denotes homotopy classes of maps)

$$[K, \varinjlim X_i] = \varinjlim [K, X_i]$$

for any compact space K. Taking K to be a sphere, one finds that the homotopy group functors commute with inductive limits of closed inclusions.

This notion is closely related to that of "smallness" introduced in **??** to prove an object can be imbedded in an injective module. For instance, smallness with respect to any limit ordinal and the class of all maps is basically equivalent to compactness in this sense.

TO BE ADDED: this should be clarified. Can we replace any inductive limit by an ordinal one, assuming there's no largest element?

2.2 Finitely presented modules

Let us recall that a module M over a ring R is said to be *finitely presented* if there is an exact sequence

$$R^m \to R^n \to M \to 0.$$

In particular, M can be described by a "finite amount of data:" M is uniquely determined by the matrix describing the map $R^m \to R^n$. Thus, to hom out of M into an R-module N is to specify the images of the n generators (that are the images of the standard basis elements in R^n), that is to pick n elements of N, and these images are required to satisfy m relations (that come from the map $R^m \to R^n$).

Note that the theory of finitely presented modules is only special and new when one works with a non-noetherian rings; over a noetherian ring, every finitely generated module is finitely presented. Nonetheless, the techniques described here are useful even if one restricts one's attention to noetherian rings.

Exercise 13.2 Show that a finitely generated *projective* module is finitely presented.

Proposition 2.3 *In the category of R-modules, the compact objects are the finitely presented ones.*

Proof. First, let us show that a finitely presented module is in fact finite. Suppose M is finitely presented and $\{N_i, i \in I\}$ is an inductive system of modules. Suppose given $M \to \varinjlim N_i$; we show that it factors through one of the N_i.

There are finitely many generators $m_1, \dots, m_n$, and in the colimit

$$N = \varinjlim N_i,$$

they must all lie in the image of some $N_j, j \in I$. Thus we can choose $r_1^{(j)}, \dots, r_n^{(j)}$ such that $r_k^{(j)}$ and m_k both map to the same thing in $\varinjlim N_i$. This alone does not enable us to conclude that $M \to \varinjlim N_i$ factors through N_j, since the relations between the $m_1, \dots, m_n$ may not be satisfied between the putative liftings $r_k^{(j)}$ to N_j.

However, we know that the relations *are* satisfied when we push down to the colimit. Since there are only finitely many relations that we need to have satisfied, we can choose $j' > j$ such that the relations all do become satisfied by the images of the $r_k^{(j)}$ in $N_{j'}$. We thus get a lifting $M \to N_{j'}$.

We see from this that the map

$$\varinjlim \mathrm{Hom}_R(M, N_i) \to \varinjlim \mathrm{Hom}_R(M, \varinjlim N_i)$$

is in fact surjective. To see that it is injective, note that if two maps $f, g : M \to N_j$ become the same map $M \to \varinjlim N_i$, then the finite set of generators $m_1, \dots, m_n$ must both be mapped to the same thing in some $N_{j'}, j' > j$.

Now suppose M is a compact object in the category of R-modules. First, we claim that M is finitely generated. Indeed, we know that M is the *inductive* limit of its finitely generated submodules. Thus we get a map

$$M \to \varinjlim_{M_F \subset M, \text{f. gen}} M_F,$$

and by hypothesis it factors as $M \to M_F$ for some M_F. This implies that $M \to M_F \to M$ is the identity, and so $M = M_F$ and M is finitely generated.

Finally, we need to see that M is finitely presented. Choose a surjection

$$R^n \twoheadrightarrow M$$

and let the kernel be K. We would like to show that K is finitely generated. Now $M \simeq R^n/K$, and consequently M is the inductive limit $\varinjlim R^n/K_F$ for K_F ranging over the finitely generated submodules of K. It follows that the natural isomorphism $M \simeq \varinjlim R^n/K_F$ factors as $M \to R^n/K_F$ for some K_F, which is thus an isomorphism. Hence M is finitely presented. ▲

The above argument shows, incidentally, that if M is finitely *generated*, then $\varinjlim \mathrm{Hom}_R(M, N_i) \to \varinjlim \mathrm{Hom}_R(M, \varinjlim N_i)$ is always *injective.*

TO BE ADDED: any module is an inductive limit of finitely presented modules **TO BE ADDED:** Lazard's theorem on flat modules

2.3 Finitely presented algebras

Let R be a commutative ring.

Definition 2.4 An R-algebra A is called **finitely presented** if A is isomorphic to an R-algebra of the form $R[x_1, \dots, x_n]/I$, where $I \subset R[x_1, \dots, x_n]$ is a finitely generated ideal in the polynomial ring. A morphism of rings $\phi : R \to R'$ is called **finitely presented** if it makes R' into a finitely presented R-algebra.

For instance, a quotient of R by a finitely generated ideal is a finitely presented R-algebra. If R is noetherian, then by the Hilbert basis theorem, an R-algebra is finitely presented if and only if it is finitely generated.

Proposition 2.5 *The finitely presented R-algebras are the compact objects in the category of R-algebras.*

We leave the proof to the reader, as it is analogous to Proposition 2.3.

The notion of a finitely presented algebra is analogous to that of a finitely presented module, insofar as a finitely presented algebra can be specified by a finite amount of "data." Namely, this data consists of the generators $x_1, \dots, x_n$ and the finitely many relations that they are required to satisfy (these finitely many relations can be taken to be generators of I). Thus, to hom out of A is "easy:" to map into an R-algebra B, we need to specify n elements of B, which have to satisfy the finitely many relations that generate the ideal I.

Like most nice types of morphisms, finitely presented morphisms have a "sorite."

Proposition 2.6 (Le sorite for finitely presented morphisms) *Finitely presented morphisms are preserved under composite and base-change. That is, if $\phi : A \to B$ is a finitely presented morphism, then:*

1. *If A' is any A-algebra, then $\phi \otimes A' : A' \to B \otimes_A A'$ is finitely presented.*
2. *If $\psi : B \to C$ is finitely presented, then C is a finitely presented over A (that is, $\psi \circ \phi$ is finitely presented).*

Proof. First, we show that finitely presented morphisms are preserved under base-change. Suppose B is finitely presented over A, thus isomorphic to a quotient $A[x_1, \dots, x_n]/I$, where I is a finitely generated ideal in the polynomial ring. Then for any A-algebra A', we have that

$$B \otimes_A A' = A'[x_1, \dots, x_n]/I'$$

where I' is the ideal in $A'[x_1, \dots, x_n]$ generated by I. (This follows by right-exactness of the tensor product.) Thus I' is finitely presented and $B \otimes_A A'$ is finitely presented over A'.

Next, we show that finitely presented morphisms are closed under composition. Suppose $A \to B$ and $B \to C$ are finitely presented morphisms. Then B is isomorphic as A-algebra to $A[x_1, \dots, x_n/I$ and C is isomorphic as B-algebra to $B[y_1, \dots, y_m]/J$, where I, J are finitely generated ideals. Thus $C \simeq A[x_1, \dots, x_n, y_1, \dots, y_m]/(I + J)$ for $I + J$ the ideal generated by I, J in $A[x_1, \dots, x_n, y_1, \dots, y_m]$. This is clearly a finitely generated ideal. ▲

Finitely presented morphisms have a curious cancellation property that we tackle next. In algebraic geometry, one often finds properties $\mathcal{P}$ of morphisms of schemes such that if a composite

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

has $\mathcal{P}$, then so does f (possibly with weak conditions on g). One example of this (in any category) is the class of monomorphisms. A more interesting example (for schemes) is the property of separatedness; the interested reader may consult [GD].

In our case, we shall illustrate this cancellation phenomenon in the category of commutative rings. Since arrows for schemes go in the opposite direction as arrows of rings, this will look slightly different.

Proposition 2.7 *Suppose we have a composite*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

such that $g \circ f : A \to C$ is finitely presented, and f is of finite type (that is, B is a finitely generated A-algebra). Then $g : B \to C$ is finitely presented.

Proof. We shall prove this using the fact that the *codiagonal* map in the category of commutative rings is finitely presented if the initial map is finitely generated:

Lemma 2.8 *Let S be a finitely generated R-algebra. Then the map $S \otimes_R S \to S$ is finitely presented.*

Proof. We shall show that the kernel I of $S \otimes_R S \to S$ is a *finitely generated* ideal. This will clearly imply the claim, as $S \otimes_R S \to S$ is obviously a surjection.

To see this, let $\alpha_1, \dots, \alpha_n \in S$ be generators for S as an R-algebra. The claim is that the elements $1 \otimes \alpha_i - \alpha_i \otimes 1$ generate I as an $S \otimes_R S$-module. Clearly these live in I. Conversely, it is clear I is generated by elements of the form $x \otimes 1 - 1 \otimes x$ (because if $z = \sum x_k \otimes y_k \in I$, then $z = \sum (x_k \otimes 1)(1 \otimes y_k - y_k \otimes y_k) + \sum x_k y_k \otimes 1$ and the last term vanishes by definition of I).

In other words, if we define $d(\alpha) = \alpha \otimes 1 - 1 \otimes \alpha$ for $\alpha \in S$, then I is generated by elements $d(\alpha)$. Now d is clearly R-linear, and we have the identity

$$\begin{aligned} d(\alpha\beta) &= \alpha\beta \otimes 1 - 1 \otimes \alpha\beta \\ &= \alpha\beta \otimes 1 - \alpha \otimes \beta + \alpha \otimes \beta - 1 \otimes \alpha\beta \\ &= (\alpha \otimes 1) d(\beta) + (1 \otimes \beta) d(\alpha). \end{aligned}$$

Thus $d(\alpha\beta)$ is in the $S \otimes_R S$-module spanned by $d(\alpha)$ and $d(\beta)$. From this, it is clear that $d(\alpha_1), d(\alpha_2), \dots, d(\alpha_n)$ generate I as a $S \otimes_R S$-module. ▲

From this lemma, we will be able to prove the theorem as follows. We can write $g : B \to C$ as the composite

$$B \to B \otimes_A C \to C$$

where the first map is the base-change of the finitely presented morphism $A \to C$ and the second morphism is the base-change of the finitely presented morphism $B \otimes_A B \to B$. Thus the composite $B \to C$ is finitely presented. ▲

§3 Inductive limits of rings

We shall now find ourselves in the following situation. We shall have an inductive system $\{A_\alpha\}_{\alpha \in I}$ of rings, indexed by a directed set I. With $A = \varinjlim A_\alpha$, we will be interested in relating categories of modules and algebras over A to the categories over A_α.

The basic idea will be as follows. Given an object (e.g. module) M of finite presentation of A, we will be able to find an object M_α of finite presentation over some A_α such that M is obtained from M_α by base-change $A_\alpha \to A$. Moreover, given a morphism $M \to N$ of objects over A, we will be able to "descend" this to a morphism $M_\alpha \to N_\alpha$ of objects of finite presentation over some A_α, which will induce $M \to N$ by base-change. In other words, the *category* of objects over A of finite presentation will be the inductive limit of the *categories* of such objects over the A_α.

3.1 Prologue: fixed points of polynomial involutions over $\mathbb{C}$

Following [Ser09], we give an application of these ideas to a simple concrete problem. This will help illustrate some of them, even though we have not formally developed the machinery yet.

If k is an algebraically closed field, a map $k^n \to k^n$ is called *polynomial* if each of the components is a polynomial function in the input coordinates. So if we identify k^n with the closed points of $\operatorname{Spec} k[x_1, \dots, x_n]$, then a polynomial function is just the restriction to to the closed points of an endomorphism of $\operatorname{Spec} k[x_1, \dots, x_n]$ induced by an algebra endomorphism.

Theorem 3.1 *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with $F \circ F = 1_{\mathbb{C}^n}$. Then F has a fixed point.*

We can phrase this alternatively as follows. Let $\sigma : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x_1, \dots, x_n]$ be a $\mathbb{C}$-involution. Then the map on the Spec's has a fixed point (which is a closed point[1]).

Proof. It is clear that the presentation of σ involves only a finite amount of data, so as in **??** we can construct a finitely generated $\mathbb{Z}$-algebra $R \subset \mathbb{C}$ and an involution

$$\overline{\sigma} : R[x_1, \dots, x_n] \to R[x_1, \dots, x_n]$$

such that σ is obtained from $\overline{\sigma}$ by base-changing $R \to \mathbb{C}$. We can assume that $\frac{1}{2} \in R$ as well. To see this explicitly, we simply need only add to R the coefficients of the polynomials $\sigma(x_1), \dots, \sigma(x_n)$, and $\frac{1}{2}$, and consider the $\mathbb{Z}$-algebra they generate.

Suppose now the system of equations $\sigma(x_1, \dots, x_n) - (x_1, \dots, x_n)$ has no solution in $\mathbb{C}^n$. This is equivalent to stating that a finite system of polynomials (namely, the $\sigma(x_i) - x_i$) generate the unit ideal in $\mathbb{C}[x_1, \dots, x_n]$, so that there are polynomials $P_i \in \mathbb{C}[x_1, \dots, x_n]$ such that $\sum P_i\left(\sigma(x_i) - x_i\right) = 1$.

Let us now enlarge R so that the coefficients of the P_i lie in R. Since the coefficients of the $\sigma(x_i)$ are already in R, we find that the polynomials $\sigma(x_i) - x_i$ will generate the unit ideal in $R[x_1, \dots, x_n]$. If R' is a homomorphic image of R, then this will be true in $R'[x_1, \dots, x_n]$.

Choose a maximal ideal $\mathfrak{m} \subset R$. Then $R/\mathfrak{m}$ is a finite field, and σ becomes an involution

$$(R/\mathfrak{m})[x_1, \dots, x_n] \to (R/\mathfrak{m})[x_1, \dots, x_n].$$

If we let $\overline{k}$ be the algebraic closure of $R/\mathfrak{m}$, then we have an involution

$$\widetilde{\sigma} : k[x_1, \dots, x_n] \to k[x_1, \dots, x_n].$$

But the induced map by $\widetilde{\sigma}$ on k^n has *no fixed points.* This follows because the $\widetilde{\sigma(x_i)} - x_i$ generate the unit ideal in $k[x_1, \dots, x_n]$ (because we can consider the images of the P_i in $k[x_1, \dots, x_n]$). Moreover, $\mathrm{char} k \neq 2$ as $\frac{1}{2} \in R$, so 2 is invertible in k as well.

[1] One can show that if there is a fixed point, there is a fixed point that is a closed point.

So from the initial fixed-point-free involution F (or σ), we have induced a polynomial map $k^n \to k^n$ with no fixed points. We need only now prove:

Lemma 3.2 *If k is the algebraic closure of $\mathbb{F}_p$ for $p \neq 2$, then any involution $F : k^n \to k^n$ which is a polynomial map has a fixed point.*

Proof. This is very simple. There is a finite field $\mathbb{F}_q$ in which the coefficients of F all lie; thus F induces a map

$$\mathbb{F}_q^n \to \mathbb{F}_q^n \qquad \blacktriangle$$

which is necessarily an involution. But an involution on a finite set of odd cardinality necessarily has a fixed point (or all orbits would be even). ▲

Remark An alternative approach to the above proof is to use a little bit of model theory. There is a general principle due to Abraham Robinson, that can be stated roughly as follows. If a sentence P in the first-order logic of fields (that is, one is allowed to refer to the elements $0, 1$ and to addition and multiplication; in addition, one is allowed to make existential and universal quantifications, negations, disjunctions, and conjunctions) has the property that P is true for an algebraically closed field of characteristic p for each $p \gg 0$, then P holds in *every* algebraically closed field of characteristic zero. This principle follows from a combination of the compactness theorem and the fact that the theory of algebraically closed fields of a fixed characteristic is *complete*: any statement is true in all of them, or in none of them.

Consider the statement $S_{n,d}$ that for any polynomial map $F : k^n \to k^n$ consisting of polynomials of degree $\leq d$ such that $F \circ F$, there is $(x_1, \dots, x_n) \in k^n$ with $F(x_1, \dots, x_n) = (x_1, \dots, x_n)$. Then $S_{n,d}$ is clearly a statement of first-order logic. Lemma 3.2 shows that $S_{n,d}$ holds in $\overline{\mathbb{F}_p}$ whenever $p > 2$. Thus, $S_{n,d}$ holds in $\mathbb{C}$ by Robinson's principle.

These types of model-theoretic arguments can be used to prove the **Ax-Grothendieck theorem**: an injective polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is surjective. See [Mar02].

3.2 The inductive limit of categories

TO BE ADDED: general formalism to clarify all this

3.3 The category of finitely presented modules

Throughout, we let $\{A_\alpha\}_{\alpha \in I}$ be an inductive system of rings, and $A = \varinjlim A_\alpha$. We are going to relate the category of finitely presented modules over A to the categories of finitely presented modules over the A_α.

We start by showing that any module over A "descends" to one of the A_α.

Proposition 3.3 *Suppose M is a finitely presented module over A. Then there is $\alpha \in I$ and a finitely presented A_α-module M_α such that $M \simeq M_\alpha \otimes_{A_\alpha} A$.*

Proof. Indeed, M is the cokernel of a morphism

$$f : A^m \to A^n$$

by definition. This morphism is described by a m-by-n (or n-by-m, depending on conventions) matrix with coefficients in A. Each of these finitely many coefficients must come from various A_α in the image (by definition of the inductive limit), and choosing α "large" we can assume that every coefficient in the matrix is in the image of $A_\alpha \to A$. Then we have a morphism

$$f_\alpha : A_\alpha^m \to A_\alpha^n$$

that induces f by base-change to A. Then we may let M_α be the cokernel of f_α since the tensor product is right-exact. ▲

Now, we want to show that if the base-change of two finitely presented modules over A_α to A become isomorphic, then they "become isomorphic" at some A_β (for $\beta > \alpha$). We shall actually prove a more general result. Namely, we shall see that a morphism at the colimit "descends" to one of the steps.

Proposition 3.4 *We keep the same notation as above. Suppose M_α, N_α are finitely presented modules over A_α. Write $M_\beta = M_\alpha \otimes_{A_\alpha} A_\beta, N_\beta = N_\alpha \otimes_{A_\alpha} A_\beta$ for each $\beta > \alpha$ and M, N for the base-changes to N.*

Suppose there is a morphism $f : M \to N$. Then there is $\beta \geq \alpha$ such that f is obtained by base-changing a morphism $f_\beta : M_\beta \to N_\beta$. If f_β, f_γ are any two morphisms that do this, then there is $\delta \geq \beta, \gamma$ such that f_β, f_γ become equal when base-changed to A_δ.

The conclusion of this result is then

$$\mathrm{Hom}_A(M, N) = \varinjlim_\beta \mathrm{Hom}_{A_\beta}(M_\beta, N_\beta).$$

The last part is essentially the "uniqueness" that we were discussing previously.

Proof. Suppose the transition maps $A_\alpha \to A_\beta$ are denoted $\phi_{\alpha\beta}$, and the natural maps $A_\alpha \to A$ are denoted ϕ_α.

We know that there are exact sequences

$$A_\alpha^m \stackrel{\mathbf{M}}{\to} A_\alpha^n \to M_\alpha \to 0,$$

and

$$A_\alpha^p \to N_\alpha \to 0.$$

These are preserved by tensoring with A. Here $\mathbf{M}$ is a suitable matrix. So we get exact sequences

$$A^m \stackrel{\phi_\alpha(\mathbf{M})}{\to} A^n \to M \to 0$$
$$A^p \to N \to 0 \qquad \blacktriangle$$

and the projectivity of A^p shows that the map $A^n \to M \to N$ can be lifted to a map $A^n \to A^p$ given by some matrix $\mathbf{M}'$ with coefficients in A. We know that there is $\mathbf{M}' \circ \phi_\alpha(\mathbf{M}) = 0$ because the map factors through M.

Now $\mathbf{M}'$ can be written as $\phi_\beta(\mathbf{M}'')$ for some matrix with coefficients in A_β, or in other words a map $A_\beta^n \to A_\beta^p$. We would like to use this to get a map $M_\beta \to A_\beta^p \to N_\beta$, but for this we need to check that $A_\beta^n \to A_\beta^p$ pulls back to zero in A_β^m. In other words, we need that $\mathbf{M}''\phi_{\alpha\beta}(\mathbf{M}) = 0$. This need not be true, but we know that it is true if base-change to a bigger β (since this matrix product is zero in the colimit). This allows us to get the map $M_\beta \to N_\beta$.

Finally, we need uniqueness. Suppose $f_\beta : M_\beta \to N_\beta$ and $f_\gamma : M_\gamma \to N_\gamma$ both are such that the base-changes to A are the same morphism $M \to N$. We need to find a δ as in the proposition. By replacing β, γ with a mutual upper bound, we may suppose that $\beta = \gamma$; we shall write the two morphisms as f_β, g_β then.

Consider the pull-backs $A_\beta^n \stackrel{f_\beta, g_\beta}{\to} N_\beta$. These uniquely determine f_β, g_β (since the map $A_\beta^n \to M_\beta$ is a surjection). These pull-backs are specified by n elements of N_β. If the base-changes of f_β, g_β via $\phi_\beta : A_\beta \to A$ are the same, then these n elements of N_β become the same in $N = \varinjlim_{\beta'} N \otimes_{A_\beta} A_{\beta'}$; thus they become equal at some finite stage, so there is $\beta' > \beta$ such that the base changes $f_{\beta'} = g_{\beta'}$.

Remark The idea of the above proof was to exploit the idea that the homomorphism carries a finite amount of data, that is the images of the generators and the condition that these images satisfy finitely many relations. In essence, it is analogous to the argument that finitely presented modules over a *fixed* ring are compact objects in that category.

Remark In fact, we can give an alternative (and slightly simpler) argument for Proposition 3.4. We know that

$$\mathrm{Hom}_{A_\beta}(M_\beta, N_\beta) = \mathrm{Hom}_{A_\alpha}(M_\alpha, N_\beta)$$

by the adjoint property of the tensor product, and similarly

$$\mathrm{Hom}_A(M, N) = \mathrm{Hom}_{A_\alpha}(M_\alpha, N).$$

So the assertion we are trying to prove is

$$\mathrm{Hom}_{A_\alpha}(M_\alpha, N) = \varinjlim_\beta \mathrm{Hom}_{A_\alpha}(M_\alpha, N_\beta),$$

which follows from Proposition 2.3.

Exercise 13.3 Give a proof of the following claim. If M is a finitely generated module over a noetherian ring R, $\mathfrak{p} \in \operatorname{Spec} R$ is such that $M_\mathfrak{p}$ is free over $R_\mathfrak{p}$, then there is $f \in R - \mathfrak{p}$ such that M_f is free over R_f.

3.4 The category of finitely presented algebras

We can treat the category of finitely presented algebras over such an inductive limit in a similar manner. As before, let $\{A_\alpha\}_{\alpha \in I}$ be an inductive system of rings with $A = \varinjlim A_\alpha$. For each α, there is a functor from the category of finitely presented A_α-algebras to the category of finitely presented A-algebras sending $C \mapsto C \otimes_{A_\alpha} A$. (Note that morphisms of finite presentation are preserved under base-change by Proposition 2.6.)

Proposition 3.5 *Suppose B is a finitely presented A-algebra. Then there is $\alpha \in I$ and a finitely presented A_α-algebra B_α such that $B \simeq B_\alpha \otimes_{A_\alpha} A$.*

Proof. This is analogous to the proof of Proposition 3.3. ▲

TO BE ADDED: analog of the next result

3.5 Spec and inductive limits

Suppose $\{A_\alpha\}_{\alpha \in I}$ is an inductive system of commutative rings, as before; we let $A = \varinjlim A_\alpha$. Since Spec is a contravariant functor, we thus find that $\operatorname{Spec} A_\alpha$ is a *projective* system of topological spaces.[2] We are now interested in relating $\operatorname{Spec} A$ to the individual $\operatorname{Spec} A_\alpha$.

Proposition 3.6 $\operatorname{Spec} A$ *is the projective limit* $\varprojlim \operatorname{Spec} A_\alpha$ *in the category of topological spaces.*

Recall that if $\{X_\alpha\}$ is a projective system of topological spaces with transition maps $\phi_{\beta\alpha} : X_\beta \to X_\alpha$ whenever $\alpha \leq \beta$, then the projective limit $\varprojlim X_\alpha$ can be constructed as follows. One considers the subset of $\prod X_\alpha$ consisting of sequences (x_α) such that $\phi_{\beta\alpha}(x_\alpha) = x_\beta$ for every $\alpha \leq \beta$. One can easily check that this has the universal property of the projective limit.

Proof. Let us first verify that the assertion is true as *sets.* There are maps

$$\operatorname{Spec} A \to \operatorname{Spec} A_\alpha$$

for each $\alpha \in I$, which are obviously compatible (since the $\{A_\alpha\}$ form an inductive system) so that they lead to a (continuous) map of topological spaces

$$\operatorname{Spec} A \to \varprojlim \operatorname{Spec} A_\alpha.$$

[2] Or schemes.

We first verify injectivity. Suppose two primes $\mathfrak{p}, \mathfrak{p}'$ were sent to the same element of $\varprojlim \operatorname{Spec} A_\alpha$. This means that if $\phi_\alpha : A_\alpha \to A$ is the natural morphism for each α, we have $\phi_\alpha^{-1}(\mathfrak{p}) = \phi_\alpha^{-1}(\mathfrak{p}')$ for all α. It follows that the intersections of $\mathfrak{p}, \mathfrak{p}'$ with the image of A_α are identical; since A is the union of $\phi_\alpha(A_\alpha)$ over all α, this implies $\mathfrak{p} = \mathfrak{p}'$.

Now let us verify surjectivity. Suppose given a sequence $\mathfrak{p}_\alpha$ of primes in A_α, for each α, such that $\mathfrak{p}_\alpha$ is the pre-image of $\mathfrak{p}_\beta$ under $A_\alpha \to A_\beta$ whenever $\alpha \leq \beta$. We want to form a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ pulling back to all these. To do this, we decide that $x \in \mathfrak{p}$ if and only if there exists $\alpha \in I$ such that $x \in \phi_\alpha(\mathfrak{p}_\alpha)$ (recall that $\phi_\alpha : A_\alpha \to A$ is the natural map). This does not depend on the choice of α, and one verifies easily that this is a prime ideal with the appropriate properties.

We now have to show that the map $\operatorname{Spec} A \to \varprojlim \operatorname{Spec} A_\alpha$ is in fact a homeomorphism. We have seen that it is continuous and bijective, so we must prove that it is open. If $a \in A$, we will be done if we can show that the image of the basic open set $D(a) \subset \operatorname{Spec} A$ is open in $\varprojlim \operatorname{Spec} A_\alpha$.

Suppose $a = \phi_\beta(a_\beta)$ for some $a_\beta \in A_\beta$. Then the claim is that the image of $D(a)$ is precisely the subset of $\varprojlim \operatorname{Spec} A_\beta$ such that the βth coordinate (which is in $\operatorname{Spec} A_\beta$!) lies in $D(a_\beta)$. This is clearly an open set, so if we prove this, then we are done. Indeed, if $\mathfrak{p} \in D(\alpha) \subset \operatorname{Spec} A$, then clearly the preimage in A_β cannot contain a_β (since a_β maps to a). Conversely, if we have a compatible sequence $\{\mathfrak{p}_\alpha\}$ of primes such that $\mathfrak{p}_\beta \in D(a_\beta)$, then the above construction of a prime $\mathfrak{p} \in \operatorname{Spec} A$ from this shows that $a \notin \mathfrak{p}$. ▲

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