Contents

Chapter 3
Three important functors

There are three functors that will be integral to our study of commutative algebra in the future: localization, the tensor product, and Hom. While localization is an *exact* functor, the tensor product and Hom are not. The failure of exactness in those cases leads to the theory of flatness and projectivity (and injectivity), and eventually the *derived functors* Tor and Ext that crop up in commutative algebra.

§1 Localization

Localization is the process of making invertible a collection of elements in a ring. It is a generalization of the process of forming a quotient field of an integral domain.

1.1 Geometric intuition

We first start off with some of the geometric intuition behind the idea of localization. Suppose we have a Riemann surface X (for example, the Riemann sphere). Let $A(U)$ be the ring of holomorphic functions over some neighborhood $U \subset X$. Now, for holomorphicity to hold, all that is required is that a function doesn't have a pole inside of U, thus when $U = X$, this condition is the strictest and as U gets smaller functions begin to show up that may not arise from the restriction of a holomorphic function over a larger domain. For example, if we want to study holomorphicity "near a point z_0" all that we should require is that the function doesn't pole at z_0. This means that we should consider quotients of holomorphic functions f/g where $g(z_0) \neq 0$. This process of inverting a collection of elements is expressed through the algebraic construction known as "localization."

1.2 Localization at a multiplicative subset

Let R be a commutative ring. We start by constructing the notion of *localization* in the most general sense.

We have already implicitly used this definition, but nonetheless, we make it formally:

Definition 1.1 A subset $S \subset R$ is a **multiplicative subset** if $1 \in S$ and if $x, y \in S$ implies $xy \in S$.

We now define the notion of *localization*. Formally, this means inverting things. This will give us a functor from R-modules to R-modules.

Definition 1.2 If M is an R-module, we define the module $S^{-1}M$ as the set of formal fractions

$$\{m/s, m \in M, s \in S\}$$

modulo an equivalence relation: where $m/s \sim m'/s'$ if and only if

$$t(s'm - m's) = 0$$

for some $t \in S$. The reason we need to include the t in the definition is that otherwise the relation would not be transitive (i.e. would not be an equivalence relation).

So two fractions agree if they agree when clearing denominators and multiplication.

It is easy to check that this is indeed an equivalence relation. Moreover $S^{-1}M$ is an abelian group with the usual addition of fractions

$$\frac{m}{s} + \frac{m'}{s'} = \frac{s'm + sm'}{ss'}$$

and it is easy to check that this is a legitimate abelian group.

Definition 1.3 Let M be an R-module and $S \subset R$ a multiplicative subset. The abelian group $S^{-1}M$ is naturally an R-module. We define

$$x(m/s) = (xm)/s, \quad x \in R.$$

It is easy to check that this is well-defined and makes it into a module.

Finally, we note that localization is a *functor* from the category of R-modules to itself. Indeed, given $f : M \to N$, there is a naturally induced map $S^{-1}M \stackrel{S^{-1}f}{\to} S^{-1}N$.

We now consider the special case when the localized module is the initial ring itself. Let $M = R$. Then $S^{-1}R$ is an R-module, and it is in fact a commutative ring in its own right. The ring structure is quite tautological:

$$(x/s)(y/s') = (xy/ss').$$

There is a map $R \to S^{-1}R$ sending $x \to x/1$, which is a ring-homomorphism.

Definition 1.4 For $S \subset R$ a multiplicative set, the localization $S^{-1}R$ is a commutative ring as above. In fact, it is an R-algebra; there is a natural map $\phi : R \to S^{-1}R$ sending $r \to r/1$.

We can, in fact, describe $\phi : R \to S^{-1}R$ by a *universal property.* Note that for each $s \in S$, $\phi(s)$ is invertible. This is because $\phi(s) = s/1$ which has a multiplicative inverse $1/s$. This property characterizes $S^{-1}R$.

For any commutative ring B, $\mathrm{Hom}(S^{-1}R, B)$ is naturally isomorphic to the subset of $\mathrm{Hom}(R, B)$ that send S to units. The map takes $S^{-1}R \to B$ to the pull-back $R \to S^{-1}R \to B$. The proof of this is very simple. Suppose that $f : R \to B$ is such that $f(s) \in B$ is invertible for each $s \in S$. Then we must define $S^{-1}R \to B$ by sending r/s to $f(r)f(s)^{-1}$. It is easy to check that this is well-defined and that the natural isomorphism as claimed is true.

Let R be a ring, M an R-module, $S \subset R$ a multiplicatively closed subset. We defined a ring of fractions $S^{-1}R$ and an R-module $S^{-1}M$. But in fact this is a module over the ring $S^{-1}R$. We just multiply $(x/t)(m/s) = (xm/st)$.

In particular, localization at S gives a *functor* from R-modules to $S^{-1}R$-modules.

Exercise 3.1 Let R be a ring, S a multiplicative subset. Let T be the R-algebra $R[\{x_s\}_{s \in S}]/(\{sx_s - 1\})$. This is the polynomial ring in the variables x_s, one for each $s \in S$, modulo the ideal generated by $sx_s = 1$. Prove that this R-algebra is naturally isomorphic to $S^{-1}R$, using the universal property.

Exercise 3.2 Define a functor **Rings** $\to$ **Sets** sending a ring to its set of units, and show that it is corepresentable (use $\mathbb{Z}[X, X^{-1}]$).

1.3 Local rings

A special case of great importance in the future is when the multiplicative subset is the complement of a prime ideal, and we study this in the present subsection. Such localizations will be "local rings" and geometrically correspond to the process of zooming at a point.

Example 1.5 Let R be an integral domain and let $S = R - \{0\}$. This is a multiplicative subset because R is a domain. In this case, $S^{-1}R$ is just the ring of fractions by allowing arbitrary nonzero denominators; it is a field, and is called the **quotient field**. The most familiar example is the construction of $\mathbb{Q}$ as the quotient field of $\mathbb{Z}$.

We'd like to generalize this example.

Example 1.6 Let R be arbitrary and $\mathfrak{p}$ is a prime ideal. This means that $1 \notin \mathfrak{p}$ and $x, y \in R - \mathfrak{p}$ implies that $xy \in R - \mathfrak{p}$. Hence, the complement $S = R - \mathfrak{p}$ is multiplicatively closed. We get a ring $S^{-1}R$.

Definition 1.7 This ring is denoted $R_{\mathfrak{p}}$ and is called the **localization at $\mathfrak{p}$.** If M is an R-module, we write $M_{\mathfrak{p}}$ for the localization of M at $R - \mathfrak{p}$.

This generalizes the previous example (where $\mathfrak{p} = (0)$).

There is a nice property of the rings $R_{\mathfrak{p}}$. To elucidate this, we start with a lemma.

Lemma 1.8 *Let R be a nonzero commutative ring. The following are equivalent:*

1. *R has a unique maximal ideal.*
2. *If $x \in R$, then either x or $1 - x$ is invertible.*

Definition 1.9 In this case, we call R **local**. A local ring is one with a unique maximal ideal.

Proof (Proof of the lemma). First we prove (2) $\implies$ (1).

Assume R is such that for each x, either x or $1-x$ is invertible. We will find the maximal ideal. Let $\mathfrak{M}$ be the collection of noninvertible elements of R. This is a subset of R, not containing 1, and it is closed under multiplication. Any proper ideal must be a subset of $\mathfrak{M}$, because otherwise that proper ideal would contain an invertible element.

We just need to check that $\mathfrak{M}$ is closed under addition. Suppose to the contrary that $x, y \in \mathfrak{M}$ but $x + y$ is invertible. We get (with $a = x/(x + y)$)

$$1 = \frac{x}{x+y} + \frac{y}{x+y} = a + (1 - a).$$

Then one of $a, 1 - a$ is invertible. So either $x(x + y)^{-1}$ or $y(x + y)^{-1}$ is invertible, which implies that either x, y is invertible, contradiction.

Now prove the reverse direction. Assume R has a unique maximal ideal $\mathfrak{M}$. We claim that $\mathfrak{M}$ consists precisely of the noninvertible elements. To see this, first note that $\mathfrak{M}$ can't contain any invertible elements since it is proper. Conversely, suppose x is not invertible, i.e. $(x) \subsetneq R$. Then (x) is contained in a maximal ideal by Proposition 4.5, so $(x) \subset \mathfrak{M}$ since $\mathfrak{M}$ is unique among maximal ideals. Thus $x \in \mathfrak{M}$.

Suppose $x \in R$; we can write $1 = x + (1 - x)$. Since $1 \notin \mathfrak{M}$, one of $x, 1 - x$ must not be in $\mathfrak{M}$, so one of those must not be invertible. So (1) $\implies$ (2). The lemma is proved. ▲

Let us give some examples of local rings.

Example 1.10 Any field is a local ring because the unique maximal ideal is (0).

Example 1.11 Let R be any commutative ring and $\mathfrak{p} \subset R$ a prime ideal. Then $R_{\mathfrak{p}}$ is a local ring. We state this as a result.

Proposition 1.12 *$R_{\mathfrak{p}}$ is a local ring if $\mathfrak{p}$ is prime.*

Proof. Let $\mathfrak{m} \subset R_{\mathfrak{p}}$ consist of elements x/s for $x \in \mathfrak{p}$ and $s \in R - \mathfrak{p}$. It is left as an exercise (using the primality of $\mathfrak{p}$) to the reader to see that whether the numerator belongs to $\mathfrak{p}$ is *independent* of the representation x/s used for it.

Then I claim that $\mathfrak{m}$ is the unique maximal ideal. First, note that $\mathfrak{m}$ is an ideal; this is evident since the numerators form an ideal. If $x/s, y/s'$ belong to $\mathfrak{m}$ with appropriate expressions, then the numerator of

$$\frac{xs' + ys}{ss'}$$

belongs to $\mathfrak{p}$, so this sum belongs to $\mathfrak{m}$. Moreover, $\mathfrak{m}$ is a proper ideal because $\frac{1}{1}$ is not of the appropriate form.

I claim that $\mathfrak{m}$ contains all other proper ideals, which will imply that it is the unique maximal ideal. Let $I \subset R_{\mathfrak{p}}$ be any proper ideal. Suppose $x/s \in I$. We want to prove $x/s \in \mathfrak{m}$. In other words, we have to show $x \in \mathfrak{p}$. But if not x/s would be invertible, and $I = (1)$, contradiction. This proves locality. ▲

Exercise 3.3 Any local ring is of the form $R_{\mathfrak{p}}$ for some ring R and for some prime ideal $\mathfrak{p} \subset R$.

Example 1.13 Let $R = \mathbb{Z}$. This is not a local ring; the maximal ideals are given by (p) for p prime. We can thus construct the localizations $\mathbb{Z}_{(p)}$ of all fractions $a/b \in \mathbb{Q}$ where $b \notin (p)$. Here $\mathbb{Z}_{(p)}$ consists of all rational numbers that don't have powers of p in the denominator.

Exercise 3.4 A local ring has no idempotents other than 0 and 1. (Recall that $e \in R$ is *idempotent* if $e^2 = e$.) In particular, the product of two rings is never local.

It may not yet be clear why localization is such a useful process. It turns out that many problems can be checked on the localizations at prime (or even maximal) ideals, so certain proofs can reduce to the case of a local ring. Let us give a small taste.

Proposition 1.14 *Let $f : M \to N$ be a homomorphism of R-modules. Then f is injective if and only if for every maximal ideal $\mathfrak{m} \subset R$, we have that $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective.*

Recall that, by definition, $M_{\mathfrak{m}}$ is the localization at $R - \mathfrak{m}$.

There are many variants on this (e.g. replace with surjectivity, bijectivity). This is a general observation that lets you reduce lots of commutative algebra to local rings, which are easier to work with.

Proof. Suppose first that each $f_{\mathfrak{m}}$ is injective. I claim that f is injective. Suppose $x \in M - \{0\}$. We must show that $f(x) \neq 0$. If $f(x) = 0$, then $f_{\mathfrak{m}}(x) = 0$ for every maximal ideal $\mathfrak{m}$. Then by injectivity it follows that x maps to zero in each $M_{\mathfrak{m}}$. We would now like to get a contradiction.

Let $I = \{a \in R : ax = 0 \in M\}$. This is proper since $x \neq 0$. So I is contained in some maximal ideal $\mathfrak{m}$. Then x maps to zero in $M_{\mathfrak{m}}$ by the previous paragraph; this means that there is $s \in R - \mathfrak{m}$ with $sx = 0 \in M$. But $s \notin I$, contradiction.

Now let us do the other direction. Suppose f is injective and $\mathfrak{m}$ a maximal ideal; we prove $f_{\mathfrak{m}}$ injective. Suppose $f_{\mathfrak{m}}(x/s) = 0 \in N_{\mathfrak{m}}$. This means that $f(x)/s = 0$ in the localized module, so that $f(x) \in M$ is killed by some $t \in R - \mathfrak{m}$. We thus have $f(tx) = t(f(x)) = 0 \in M$. This means that $tx = 0 \in M$ since f is injective. But this in turn means that $x/s = 0 \in M_{\mathfrak{m}}$. This is what we wanted to show. ▲

1.4 Localization is exact

Localization is to be thought of as a very mild procedure.

The next result says how inoffensive localization is. This result is a key tool in reducing problems to the local case.

Proposition 1.15 *Suppose $f : M \to N, g : N \to P$ and $M \to N \to P$ is exact. Let $S \subset R$ be multiplicatively closed. Then*

$$S^{-1}M \to S^{-1}N \to S^{-1}P$$

is exact.

Or, as one can alternatively express it, localization is an *exact functor.*

Before proving it, we note a few corollaries:

Corollary 1.16 *If $f : M \to N$ is surjective, then $S^{-1}M \to S^{-1}N$ is too.*

Proof. To say that $A \to B$ is surjective is the same as saying that $A \to B \to 0$ is exact. From this the corollary is evident. ▲

Similarly:

Corollary 1.17 *If $f : M \to N$ is injective, then $S^{-1}M \to S^{-1}N$ is too.*

Proof. To say that $A \to B$ is injective is the same as saying that $0 \to A \to B$ is exact. From this the corollary is evident. ▲

Proof (Proof of the proposition). We adopt the notation of the proposition. If the composite $g \circ f$ is zero, clearly the localization $S^{-1}M \to S^{-1}N \to S^{-1}P$ is zero too. Call the maps $S^{-1}M \to S^{-1}N, S^{-1}N \to S^{-1}P$ as ϕ, ψ. We know that $\psi \circ \phi = 0$ so $\ker(\psi) \supset \mathrm{Im}(\phi)$. Conversely, suppose something belongs to $\ker(\psi)$. This can be written as a fraction

$$x/s \in \ker(\psi)$$

where $x \in N, s \in S$. This is mapped to

$$g(x)/s \in S^{-1}P,$$

which we're assuming is zero. This means that there is $t \in S$ with $tg(x) = 0 \in P$. This means that $g(tx) = 0$ as an element of P. But $tx \in N$ and its image of g vanishes, so tx must come from something in M. In particular,

$$tx = f(y) \text{ for some } y \in M.$$

In particular,

$$\frac{x}{s} = \frac{tx}{ts} = \frac{f(y)}{ts} = \phi(y/ts) \in \mathrm{Im}(\phi).$$

This proves that anything belonging to the kernel of ψ lies in $\mathrm{Im}(\phi)$. ▲

1.5 Nakayama's lemma

We now state a very useful criterion for determining when a module over a *local* ring is zero.

Lemma 1.18 (Nakayama's lemma) *If R is a local ring with maximal ideal $\mathfrak{m}$. Let M be a finitely generated R-module. If $\mathfrak{m}M = M$, then $M = 0$.*

Note that $\mathfrak{m}M$ is the submodule generated by products of elements of $\mathfrak{m}$ and M.

Remark Once one has the theory of the tensor product, this equivalently states that if M is finitely generated, then

$$M \otimes_R R/\mathfrak{m} = M/\mathfrak{m}M \neq 0.$$

So to prove that a finitely generated module over a local ring is zero, you can reduce to studying the reduction to $R/\mathfrak{m}$. This is thus a very useful criterion.

Nakayama's lemma highlights why it is so useful to work over a local ring. Thus, it is useful to reduce questions about general rings to questions about local rings. Before proving it, we note a corollary.

Corollary 1.19 *Let R be a local ring with maximal ideal $\mathfrak{m}$, and M a finitely generated module. If $N \subset M$ is a submodule such that $N + \mathfrak{m}N = M$, then $N = M$.*

Proof. Apply Nakayama above (Lemma 1.18) to M/N. ▲

We shall prove more generally:

Proposition 1.20 *Suppose M is a finitely generated R-module, $J \subset R$ an ideal. Suppose $JM = M$. Then there is $a \in 1 + J$ such that $aM = 0$.*

If J is the maximal ideal of a local ring, then a is a unit, so that $M = 0$.

Proof. Suppose M is generated by $\{x_1, \dots, x_n\} \subset M$. This means that every element of M is a linear combination of elements of x_i. However, each $x_i \in JM$ by assumption. In particular, each x_i can be written as

$$x_i = \sum a_{ij} x_j, \text{ where } a_{ij} \in \mathfrak{m}.$$

If we let A be the matrix $\{a_{ij}\}$, then A sends the vector (x_i) into itself. In particular, $I - A$ kills the vector (x_i).

Now $I - A$ is an n-by-n matrix in the ring R. We could, of course, reduce everything modulo J to get the identity; this is because A consists of elements of J. It follows that the determinant must be congruent to 1 modulo J.

In particular, $a = \det(I - A)$ lies in $1 + J$. Now by familiar linear algebra, aI can be represented as the product of A and the matrix of cofactors; in particular, aI annihilates the vector (x_i), so that $aM = 0$. ▲

Before returning to the special case of local rings, we observe the following useful fact from ideal theory:

Proposition 1.21 *Let R be a commutative ring, $I \subset R$ a finitely generated ideal such that $I^2 = I$. Then I is generated by an idempotent element.*

Proof. We know that there is $x \in 1 + I$ such that $xI = 0$. If $x = 1 + y, y \in I$, it follows that

$$yt = t$$

for all $t \in I$. In particular, y is idempotent and $(y) = I$. ▲

Exercise 3.5 Proposition 1.21 fails if the ideal is not finitely generated.

Exercise 3.6 Let M be a finitely generated module over a ring R. Suppose $f : M \to M$ is a surjection. Then f is an isomorphism. To see this, consider M as a module over $R[t]$ with t acting by f; since $(t)M = M$, argue that there is a polynomial $Q(t) \in R[t]$ such that $Q(t)t$ acts as the identity on M, i.e. $Q(f)f = 1_M$.

Exercise 3.7 Give a counterexample to the conclusion of Nakayama's lemma when the module is not finitely generated.

Exercise 3.8 Let M be a finitely generated module over the ring R. Let $\mathfrak{J}$ be the Jacobson radical of R (cf. **??** 1.26). If $\mathfrak{J}M = M$, then $M = 0$.

Exercise 3.9 (A converse to Nakayama's lemma) Suppose conversely that R is a ring, and $\mathfrak{a} \subset R$ an ideal such that $\mathfrak{a}M \neq M$ for every nonzero finitely generated R-module. Then $\mathfrak{a}$ is contained in every maximal ideal of R.

Exercise 3.10 Here is an alternative proof of Nakayama's lemma. Let R be local with maximal ideal $\mathfrak{m}$, and let M be a finitely generated module with $\mathfrak{m}M = M$. Let n be the minimal number of generators for M. If $n > 0$, pick generators $x_1, \dots, x_n$. Then write $x_1 = a_1x_1 + \dots + a_nx_n$ where each $a_i \in \mathfrak{m}$. Deduce that x_1 is in the submodule generated by the $x_i, i \geq 2$, so that n was not actually minimal, contradiction.

Let M, M' be finitely generated modules over a local ring $(R, \mathfrak{m})$, and let $\phi : M \to M'$ be a homomorphism of modules. Then Nakayama's lemma gives a criterion for ϕ to be a surjection: namely, the map $\overline{\phi} : M/\mathfrak{m}M \to M'/\mathfrak{m}M'$ must be a surjection. For injections, this is false. For instance, if ϕ is multiplication by any element of $\mathfrak{m}$, then $\overline{\phi}$ is zero but ϕ may yet be injective. Nonetheless, we give a criterion for a map of *free* modules over a local ring to be a *split* injection.

Proposition 1.22 *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let F, F' be two finitely generated free R-modules, and let $\phi : F \to F'$ be a homomorphism. Then ϕ is a split injection if and only if the reduction $\overline{\phi}$*

$$F/\mathfrak{m}F \stackrel{\overline{\phi}}{\to} F'/\mathfrak{m}F'$$

is an injection.

Proof. One direction is easy. If ϕ is a split injection, then it has a left inverse $\psi : F' \to F$ such that $\psi \circ \phi = 1_F$. The reduction of ψ as a map $F'/\mathfrak{m}F' \to F/\mathfrak{m}F$ is a left inverse to $\overline{\phi}$, which is thus injective.

Conversely, suppose $\overline{\phi}$ injective. Let $e_1, \dots, e_r$ be a "basis" for F, and let $f_1, \dots, f_r$ be the images under ϕ in F'. Then the reductions $\overline{f_1}, \dots, \overline{f_r}$ are linearly independent in the $R/\mathfrak{m}$-vector space $F'/\mathfrak{m}F'$. Let us complete this to a basis of $F'/\mathfrak{m}F'$ by adding elements $\overline{g_1}, \dots, \overline{g_s} \in F'/\mathfrak{m}F'$, which we can lift to elements $g_1, \dots, g_s \in F'$. It is clear that F' has rank $r + s$ since its reduction $F'/\mathfrak{m}F'$ does.

We claim that the set $\{f_1, \dots, f_r, g_1, \dots, g_s\}$ is a basis for F'. Indeed, we have a map

$$R^{r+s} \to F'$$

of free modules of rank $r + s$. It can be expressed as an $r + s$-by-$r + s$ matrix M; we need to show that M is invertible. But if we reduce modulo $\mathfrak{m}$, it is invertible since the reductions of $f_1, \dots, f_r, g_1, \dots, g_s$ form a basis of $F'/\mathfrak{m}F'$. Thus the determinant of M is not in $\mathfrak{m}$, so by locality it is invertible. The claim about F' is thus proved.

We can now define the left inverse $F' \to F$ of ϕ. Indeed, given $x \in F'$, we can write it uniquely as a linear combination $\sum a_i f_i + \sum b_j g_j$ by the above. We define $\psi(\sum a_i f_i + \sum b_j g_j) = \sum a_i e_i \in F$. It is clear that this is a left inverse ▲

We next note a slight strenghtening of the above result, which is sometimes useful. Namely, the first module does not have to be free.

Proposition 1.23 *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let M, F be two finitely generated R-modules with F free, and let $\phi : M \to F'$ be a homomorphism. Then ϕ is a split injection if and only if the reduction $\overline{\phi}$*

$$M/\mathfrak{m}M \stackrel{\overline{\phi}}{\to} F/\mathfrak{m}F$$

is an injection.

It will in fact follow that M is itself free, because M is projective (see **??** below) as it is a direct summand of a free module.

Proof. Let L be a "free approximation" to M. That is, choose a basis $\overline{x_1}, \dots, \overline{x_n}$ for $M/\mathfrak{m}M$ (as an $R/\mathfrak{m}$-vector space) and lift this to elements $x_1, \dots, x_n \in M$. Define a map

$$L = R^n \to M$$

by sending the ith basis vector to x_i. Then $L/\mathfrak{m}L \to M/\mathfrak{m}M$ is an isomorphism. By Nakayama's lemma, $L \to M$ is surjective.

Then the composite map $L \to M \to F$ is such that the $L/\mathfrak{m}L \to F/\mathfrak{m}F$ is injective, so $L \to F$ is a split injection (by Proposition 1.22). It follows that we can find a splitting $F \to L$, which when composed with $L \to M$ is a splitting of $M \to F$. ▲

Exercise 3.11 Let A be a local ring, and B a ring which is finitely generated and free as an A-module. Suppose $A \to B$ is an injection. Then $A \to B$ is a *split injection.* (Note that any nonzero morphism mapping out of a field is injective.)

§2 The functor Hom

In any category, the morphisms between two objects form a set.[1] In many categories, however, the hom-sets have additional structure. For instance, the hom-sets between abelian groups are themselves abelian groups. The same situation holds for the category of modules over a commutative ring.

Definition 2.1 Let R be a commutative ring and M, N to be R-modules. We write $\mathrm{Hom}_R(M, N)$ for the set of all R-module homomorphisms $M \to N$. $\mathrm{Hom}_R(M, N)$ is an R-module because one can add homomorphisms $f, g : M \to N$ by adding them pointwise: if f, g are homomorphisms $M \to N$, define $f + g : M \to N$ via $(f + g)(m) = f(m) + g(m)$; similarly, one can multiply homomorphisms $f : M \to N$ by elements $a \in R$: one sets $(af)(m) = a(f(m))$.

Recall that in any category, the hom-sets are *functorial.* For instance, given $f : N \to N'$, post-composition with f defines a map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N')$ for any M. Similarly precomposition gives a natural map $\mathrm{Hom}_R(N', M) \to \mathrm{Hom}_R(N, M)$. In particular, we get a bifunctor Hom, contravariant in the first variable and covariant in the second, of R-modules into R-modules.

2.1 Left-exactness of Hom

We now discuss the exactness properties of this construction of forming Hom-sets. The following result is basic and is, in fact, a reflection of the universal property of the kernel.

Proposition 2.2 *If M is an R-module, then the functor*

$$N \to \mathrm{Hom}_R(M, N)$$

is left exact (but not exact *in general).*

[1] Strictly speaking, this may depend on your set-theoretic foundations.

This means that if

$$0 \to N' \to N \to N''$$

is exact, then

$$0 \to \mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'')$$

is exact as well.

Proof. First, we have to show that the map $\mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N)$ is injective; this is because $N' \to N$ is injective, and composition with $N' \to N$ can't kill any nonzero $M \to N'$. Similarly, exactness in the middle can be checked easily, and follows from **??** 1.17; it states simply that a map $M \to N$ has image landing inside N' (i.e. factors through N') if and only if it composes to zero in N''. ▲

This functor $\mathrm{Hom}_R(M, \cdot)$ is not exact in general. Indeed:

Example 2.3 Suppose $R = \mathbb{Z}$, and consider the R-module (i.e. abelian group) $M = \mathbb{Z}/2\mathbb{Z}$. There is a short exact sequence

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Let us apply $\mathrm{Hom}_R(M, \cdot)$. We get a *complex*

$$0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, 2\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to 0.$$

The second-to-last term is $\mathbb{Z}/2\mathbb{Z}$; everything else is zero. Thus the sequence is not exact, and in particular the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ is not an exact functor.

We have seen that homming out of a module is left-exact. Now, we see the same for homming *into* a module.

Proposition 2.4 *If M is a module, then $\mathrm{Hom}_R(-, M)$ is a left-exact contravariant functor.*

We write this proof in slightly more detail than Proposition 2.2, because of the contravariance.

Proof. We want to show that $\mathrm{Hom}(\cdot, M)$ is a left-exact contravariant functor, which means that if $A \xrightarrow{u} B \xrightarrow{v} C \to 0$ is exact, then so is

$$0 \to \mathrm{Hom}(C, M) \xrightarrow{\mathbf{v}} \mathrm{Hom}(B, M) \xrightarrow{\mathbf{u}} \mathrm{Hom}(A, M)$$

is exact. Here, the bold notation refers to the induced maps of u, v on the hom-sets: if $f \in \mathrm{Hom}(B, M)$ and $g \in \mathrm{Hom}(C, M)$, we define $\mathbf{u}$ and $\mathbf{v}$ via $\mathbf{v}(g) = g \circ v$ and $\mathbf{u}(f) = f \circ u$.

Let us show first that $\mathbf{v}$ is injective. Suppose that $g \in \mathrm{Hom}(C, M)$. If $\mathbf{v}(g) = g \circ v = 0$ then $(g \circ v)(b) = 0$ for all $b \in B$. Since v is a surjection, this means that $g(C) = 0$ and hence $g = 0$. Therefore, $\mathbf{v}$ is injective, and we have exactness at $\mathrm{Hom}(C, M)$.

Since $v \circ u = 0$, it is clear that $\mathbf{u} \circ \mathbf{u} = 0$.

Now, suppose that $f \in \ker(\mathbf{u}) \subset \mathrm{Hom}(B, M)$. Then $\mathbf{u}(f) = f \circ u = 0$. Thus $f : B \to M$ factors through $B/\mathrm{Im}(u)$. However, $\mathrm{Im}(u) = \ker(v)$, so f factors through $B/\ker(v)$. Exactness shows that there is an isomorphism $B/\ker(v) \simeq C$. In particular, we find that f factors through C. This is what we wanted. ▲

Exercise 3.12 Come up with an example where $\mathrm{Hom}_R(-, M)$ is not exact.

Exercise 3.13 Over a *field*, Hom is always exact.

2.2 Projective modules

Let M be an R-module for a fixed commutative ring R. We have seen that $\mathrm{Hom}_R(M, -)$ is generally only a left-exact functor. Sometimes, however, we do have exactness. We axiomatize this with the following.

Definition 2.5 An R-module M is called **projective** if the functor $\mathrm{Hom}_R(M, \cdot)$ is exact.[2]

One may first observe that a free module is projective. Indeed, let $F = R^I$ for an indexing set. Then the functor $N \to \mathrm{Hom}_R(F, N)$ is naturally isomorphic to $N \to N^I$. It is easy to see that this functor preserves exact sequences (that is, if $0 \to A \to B \to C \to 0$ is exact, so is $0 \to A^I \to B^I \to C^I \to 0$). Thus F is projective. One can also easily check that a *direct summand* of a projective module is projective.

It turns out that projective modules have a very clean characterization. They are *precisely* the direct summands in free modules.

TO BE ADDED: check this

Proposition 2.6 *The following are equivalent for an R-module M:*

1. *M is projective.*
2. *Given any map $M \to N/N'$ from M into a quotient of R-module N/N', we can lift it to a map $M \to N$.*
3. *There is a module M' such that $M \oplus M'$ is free.*

Proof. The equivalence of 1 and 2 is just unwinding the definition of projectivity, because we just need to show that $\mathrm{Hom}_R(M, \cdot)$ preserves surjective maps, i.e. quotients. ($\mathrm{Hom}_R(M, \cdot)$ is already left-exact, after all.) To say that $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N/N')$ is surjective is just the statement that any map $M \to N/N'$ can be lifted to $M \to N$.

Let us show that 2 implies 3. Suppose M satisfies 2. Then choose a surjection $P \twoheadrightarrow M$ where P is free, by **??**. Then we can write $M \simeq P/P'$ for a submodule $P' \subset P$. The isomorphism map $M \to P/P'$ leads by 2 to a lifting $M \to P$. In particular, there is a section of $P \to M$, namely this lifting. Since a section leads to a split exact sequence by **??**, we find then that $P \simeq \ker(P \to M) \oplus \mathrm{Im}(M \to P) \simeq \ker(P \to M) \oplus M$, verifying 3 since P is free.

Now let us show that 3 implies 2. Suppose $M \oplus M'$ is free, isomorphic to P. Then a map $M \to N/N'$ can be extended to

$$P \to N/N'$$

by declaring it to be trivial on M'. But now $P \to N/N'$ can be lifted to N because P is free, and we have observed that a free module is projective above; alternatively, we just lift the image of a basis. This defines $P \to N$. We may then compose this with the inclusion $M \to P$ to get the desired map $M \to P \to N$, which is a lifting of $M \to N/N'$. ▲

Of course, the lifting $P \to N$ of a given map $P \to N/N'$ is generally not unique, and in fact is unique precisely when $\mathrm{Hom}_R(P, N') = 0$.

So projective modules are precisely those with the following lifting property. Consider a diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

[2]It is possible to define a projective module over a noncommutative ring. The definition is the same, except that the Hom-sets are no longer modules, but simply abelian groups.

where the bottom row is exact. Then, if P is projective, there is a lifting $P \to M$ making commutative the diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \downarrow & & \\ M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

Corollary 2.7 *Let M be a module. Then there is a surjection $P \twoheadrightarrow M$, where P is projective.*

Proof. Indeed, we know (Proposition 6.6) that we can always get a surjection from a free module. Since free modules are projective by Proposition 2.6, we are done. ▲

Exercise 3.14 Let R be a principal ideal domain, F' a submodule of a free module F. Show that F' is free. (Hint: well-order the set of generators of F, and climb up by transfinite induction.) In particular, any projective modules is free.

2.3 Example: the Serre-Swan theorem

We now briefly digress to describe an important correspondence between projective modules and vector bundles. The material in this section will not be used in the sequel.

Let X be a compact space. We shall not recall the topological notion of a *vector bundle* here.

We note, however, that if E is a (complex) vector bundle, then the set $\Gamma(X, E)$ of global sections is naturally a module over the ring $C(X)$ of complex-valued continuous functions on X.

Proposition 2.8 *If E is a vector bundle on a compact Hausdorff space X, then there is a surjection $\mathcal{O}^N \twoheadrightarrow E$ for some N.*

Here $\mathcal{O}^N$ denotes the trivial bundle.

It is known that in the category of vector bundles, every epimorphism splits. In particular, it follows that E can be viewed as a *direct summand* of the bundle $\mathcal{O}^N$. Since $\Gamma(X, E)$ is then a direct summand of $\Gamma(X, \mathcal{O}^N) = C(X)^N$, we find that $\Gamma(X, E)$ is a direct summand of a projective $C(X)$-module. Thus:

Proposition 2.9 $\Gamma(X, E)$ *is a finitely generated projective $C(X)$-module.*

Theorem 2.10 (Serre-Swan) *The functor $E \mapsto \Gamma(X, E)$ induces an equivalence of categories between vector bundles on X and finitely generated projective modules over $C(X)$.*

2.4 Injective modules

We have given a complete answer to the question of when the functor $\mathrm{Hom}_R(M, -)$ is exact. We have shown that there are a lot of such *projective* modules in the category of R-modules, enough that any module admits a surjection from one such. However, we now have to answer the dual question: when is the functor $\mathrm{Hom}_R(-, Q)$ exact?

Let us make the dual definition:

Definition 2.11 An R-module Q is **injective** if the functor $\mathrm{Hom}_R(-, Q)$ is exact.

Thus, a module Q over a ring R is injective if whenever $M \to N$ is an injection, and one has a map $M \to Q$, it can be extended to $N \to Q$: in other words, $\mathrm{Hom}_R(N, Q) \to \mathrm{Hom}_R(M, Q)$ is surjective. We can visualize this by a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \longrightarrow & N \\ & & \downarrow & \swarrow & \\ & & Q & & \end{array}$$

where the dotted arrow always exists if Q is injective.

The notion is dual to projectivity, in some sense, so just as every module M admits an epimorphic map $P \to M$ for P projective, we expect by duality that every module admits a monomorphic map $M \to Q$ for Q injective. This is in fact true, but will require some work. We start, first, with a fact about injective abelian groups.

Theorem 2.12 *A divisible abelian group (i.e. one where the map $x \to nx$ for any $n \in \mathbb{N}$ is surjective) is injective as a $\mathbb{Z}$-module (i.e. abelian group).*

Proof. The actual idea of the proof is rather simple, and similar to the proof of the Hahn-Banach theorem. Namely, we extend bit by bit, and then use Zorn's lemma.

The first step is that we have a subgroup M of a larger abelian group N. We have a map of $f : M \to Q$ for Q some divisible abelian group, and we want to extend it to N.

Now we can consider the poset of pairs $(\tilde{f}, M')$ where $M' \supset M$, and $\tilde{f} : M' \to N$ is a map extending f. Naturally, we make this into a poset by defining the order as "$(\tilde{f}, M') \leq (\tilde{f}', M'')$ if M'' contains M' and $\tilde{f}'$ is an extension of $\tilde{f}$. It is clear that every chain has an upper bound, so Zorn's lemma implies that we have a submodule $M' \subset N$ containing M, and a map $\tilde{f} : M' \to N$ extending f, such that there is no proper extension of $\tilde{f}$. From this we will derive a contradiction unless $M' = N$.

So suppose we have $M' \neq N$, for M' the maximal submodule to which f can be extended, as in the above paragraph. Pick $m \in N - M'$, and consider the submodule $M' + \mathbb{Z}m \subset N$. We are going to show how to extend $\tilde{f}$ to this bigger submodule. First, suppose $\mathbb{Z}m \cap M' = \{0\}$, i.e. the sum is direct. Then we can extend $\tilde{f}$ because $M' + \mathbb{Z}m$ is a direct sum: just define it to be zero on $\mathbb{Z}m$.

The slightly harder part is what happens if $\mathbb{Z}m \cap M' \neq \{0\}$. In this case, there is an ideal $I \subset \mathbb{Z}$ such that $n \in I$ if and only if $nm \in M'$. This ideal, however, is principal; let $g \in \mathbb{Z} - \{0\}$ be a generator. Then $gm = p \in M'$. In particular, $\tilde{f}(gm)$ is defined. We can "divide" this by g, i.e. find $u \in Q$ such that $gu = \tilde{f}(gm)$.

Now we may extend to a map $\tilde{f}'$ from $\mathbb{Z}m + M'$ into Q as follows. Choose $m' \in M', k \in \mathbb{Z}$. Define $\tilde{f}'(m' + km) = \tilde{f}(m') + ku$. It is easy to see that this is well-defined by the choice of u, and gives a proper extension of $\tilde{f}$. This contradicts maximality of M' and completes the proof. ▲

Exercise 3.15 Theorem 2.12 works over any principal ideal domain.

Exercise 3.16 (Baer) Let N be an R-module such that for any ideal $I \subset R$, any morphism $I \to N$ can be extended to $R \to N$. Then N is injective. (Imitate the above argument.)

From this, we may prove:

Theorem 2.13 *Any R-module M can be imbedded in an injective R-module Q.*

Proof. First of all, we know that any R-module M is a quotient of a free R-module. We are going to show that the dual (to be defined shortly) of a free module is injective. And so since every module admits a surjection from a free module, we will use a dualization argument to prove the present theorem.

First, for any abelian group G, define the **dual group** as $G^\vee = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$. Dualization is clearly a contravariant functor from abelian groups to abelian groups. By Proposition 2.4 and Theorem 2.12, an exact sequence of groups

$$0 \to A \to B \to C \to 0$$

induces an exact sequence

$$0 \to C^\vee \to B^\vee \to A^\vee \to 0.$$

In particular, dualization is an exact functor:

Proposition 2.14 *Dualization preserves exact sequences (but reverses the order).*

Now, we are going to apply this to R-modules. The dual of a left R-module is acted upon by R. The action, which is natural enough, is as follows. Let M be an R-module, and $f : M \to \mathbb{Q}/\mathbb{Z}$ be a homomorphism of abelian groups (since $\mathbb{Q}/\mathbb{Z}$ has in general no R-module structure), and $r \in R$; then we define rf to be the map $M \to \mathbb{Q}/\mathbb{Z}$ defined via

$$(rf)(m) = f(rm).$$

It is easy to check that $M^\vee$ is thus made into an R-module.[3] In particular, dualization into $\mathbb{Q}/\mathbb{Z}$ gives a contravariant exact functor from *R-modules* to *R-modules*.

Let M be as before, and now consider the R-module $M^\vee$. By **??**, we can find a free module F and a surjection

$$F \to M^\vee \to 0.$$

Now dualizing gives an exact sequence of R-modules

$$0 \to M^{\vee\vee} \to F^\vee.$$

However, there is a natural map (of R-modules) $M \to M^{\vee\vee}$: given $m \in M$, we can define a functional $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ by evaluation at m. One can check that this is a homomorphism. Moreover, this morphism $M \to M^{\vee\vee}$ is actually injective: if $m \in M$ were in the kernel, then by definition every functional $M \to \mathbb{Q}/\mathbb{Z}$ must vanish on m. It is easy to see (using $\mathbb{Z}$-injectivity of $\mathbb{Q}/\mathbb{Z}$) that this cannot happen if $m \neq 0$: we could just pick a nontrivial functional on the monogenic *subgroup* $\mathbb{Z}m$ and extend to M.

We claim now that $F^\vee$ is injective. This will prove the theorem, as we have the composite of monomorphisms $M \hookrightarrow M^{\vee\vee} \hookrightarrow F^\vee$ that embeds M inside an injective module.

Lemma 2.15 *The dual of a free R-module F is an injective R-module.*

Proof. Let $0 \to A \to B$ be exact; we have to show that

$$\mathrm{Hom}_R(B, F^\vee) \to \mathrm{Hom}_R(A, F^\vee) \to 0.$$

is exact. Now we can reduce to the case where F is the R-module R itself. Indeed, F is a direct sum of R's by assumption, and taking hom's turns them into direct products; moreover the direct product of exact sequences is exact.

So we are reduced to showing that $R^\vee$ is injective. Now we claim that

$$\mathrm{Hom}_R(B, R^\vee) = \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}). \tag{3.1}$$

▲

In particular, $\mathrm{Hom}_R(-, R^\vee)$ is an exact functor because $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group. The proof of Eq. (3.1) is actually "trivial." For instance, a R-homomorphism $f : B \to R^\vee$ induces $\tilde{f} : B \to \mathbb{Q}/\mathbb{Z}$ by sending $b \to (f(b))(1)$. One checks that this is bijective.

[3] If R is noncommutative, this would not work: instead $M^\vee$ would be an *right* R-module. For commutative rings, we have no such distinction between left and right modules.

2.5 The small object argument

There is another, more set-theoretic approach to showing that any R-module M can be imbedded in an injective module. This approach, which constructs the injective module by a transfinite colimit of push-outs, is essentially analogous to the "small object argument" that one uses in homotopy theory to show that certain categories (e.g. the category of CW complexes) are model categories in the sense of Quillen; see [Hov07]. While this method is somewhat abstract and more complicated than the one of Section 2.4, it is also more general. Apparently this method originates with Baer, and was revisited by Cartan and Eilenberg in [**?**] and by Grothendieck in [Gro57]. There Grothendieck uses it to show that many other abelian categories have enough injectives.

We first begin with a few remarks on smallness. Let $\{B_\alpha\}, \alpha \in \mathcal{A}$ be an inductive system of objects in some category $\mathcal{C}$, indexed by an ordinal $\mathcal{A}$. Let us assume that $\mathcal{C}$ has (small) colimits. If A is an object of $\mathcal{C}$, then there is a natural map

$$\varinjlim \operatorname{Hom}(A, B_\alpha) \to \operatorname{Hom}(A, \varinjlim B_\alpha) \tag{3.2}$$

because if one is given a map $A \to B_\beta$ for some β, one naturally gets a map from A into the colimit by composing with $B_\beta \to \varinjlim B_\alpha$. (Note that the left colimit is one of sets!)

In general, the map Eq. (3.2) is neither injective or surjective.

Example 2.16 Consider the category of sets. Let $A = \mathbb{N}$ and $B_n = \{1, \dots, n\}$ be the inductive system indexed by the natural numbers (where $B_n \to B_m, n \leq m$ is the obvious map). Then $\varinjlim B_n = \mathbb{N}$, so there is a map

$$A \to \varinjlim B_n,$$

which does not factor as

$$A \to B_m$$

for any m. Consequently, $\varinjlim \operatorname{Hom}(A, B_n) \to \operatorname{Hom}(A, \varinjlim B_n)$ is not surjective.

Example 2.17 Next we give an example where the map fails to be injective. Let $B_n = \mathbb{N}/\{1, 2, \dots, n\}$, that is, the quotient set of $\mathbb{N}$ with the first n elements collapsed to one element. There are natural maps $B_n \to B_m$ for $n \leq m$, so the $\{B_n\}$ form an inductive system. It is easy to see that the colimit $\varinjlim B_n = \{*\}$: it is the one-point set. So it follows that $\operatorname{Hom}(A, \varinjlim B_n)$ is a one-element set.

However, $\varinjlim \operatorname{Hom}(A, B_n)$ is *not* a one-element set. Consider the family of maps $A \to B_n$ which are just the natural projections $\mathbb{N} \to \mathbb{N}/\{1, 2, \dots, n\}$ and the family of maps $A \to B_n$ which map the whole of A to the class of 1. These two families of maps are distinct at each step and thus are distinct in $\varinjlim \operatorname{Hom}(A, B_n)$, but they induce the same map $A \to \varinjlim B_n$.

Nonetheless, if A is a *finite set*, it is easy to see that for any sequence of sets $B_1 \to B_2 \to \dots$, we have

$$\varinjlim \operatorname{Hom}(A, B_n) = \operatorname{Hom}(A, \varinjlim B_n).$$

Proof. Let $f : A \to \varinjlim B_n$. The range of A is finite, containing say elements $c_1, \dots, c_r \in \varinjlim B_n$. These all come from some elements in B_N for N large by definition of the colimit. Thus we can define $\widetilde{f} : A \to B_N$ lifting f at a finite stage.

Next, suppose two maps $f_n : A \to B_m, g_n : A \to B_m$ define the same map $A \to \varinjlim B_n$. Then each of the finitely many elements of A gets sent to the same point in the colimit. By definition of the colimit for sets, there is $N \geq m$ such that the finitely many elements of A get sent to the same points in B_N under f and g. This shows that $\varinjlim \operatorname{Hom}(A, B_n) \to \operatorname{Hom}(A, \varinjlim B_n)$ is injective. ▲

The essential idea is that A is "small" relative to the long chain of compositions $B_1 \to B_2 \to \dots$, so that it has to factor through a finite step.

Let us generalize this.

Definition 2.18 Let $\mathcal{C}$ be a category, I a class of maps, and ω an ordinal. An object $A \in \mathcal{C}$ is said to be ω-**small** (with respect to I) if whenever $\{B_\alpha\}$ is an inductive system parametrized by ω with maps in I, then the map

$$\varinjlim \operatorname{Hom}(A, B_\alpha) \to \operatorname{Hom}(A, \varinjlim B_\alpha)$$

is an isomorphism.

Our definition varies slightly from that of [Hov07], where only "nice" transfinite sequences $\{B_\alpha\}$ are considered.

In our applications, we shall begin by restricting ourselves to the category of R-modules for a fixed commutative ring R. We shall also take I to be the set of *monomorphisms,* or injections.[4] Then each of the maps

$$B_\beta \to \varinjlim B_\alpha$$

is an injection, so it follows that $\operatorname{Hom}(A, B_\beta) \to \operatorname{Hom}(A, \varinjlim B_\alpha)$ is one, and in particular the canonical map

$$\varinjlim \operatorname{Hom}(A, B_\alpha) \to \operatorname{Hom}(A, \varinjlim B_\alpha) \tag{3.3}$$

is an *injection.* We can in fact interpret the B_α's as subobjects of the big module $\varinjlim B_\alpha$, and think of their union as $\varinjlim B_\alpha$. (This is not an abuse of notation if we identify B_α with the image in the colimit.)

We now want to show that modules are always small for "large" ordinals ω. For this, we have to digress to do some set theory:

Definition 2.19 Let ω be a *limit* ordinal, and κ a cardinal. Then ω is κ-**filtered** if every collection C of ordinals strictly less than ω and of cardinality at most κ has an upper bound strictly less than ω.

Example 2.20 A limit ordinal (e.g. the natural numbers ω_0) is κ-filtered for any finite cardinal κ.

Proposition 2.21 *Let κ be a cardinal. Then there exists a κ-filtered ordinal ω.*

Proof. If κ is finite, Example 2.20 shows that any limit ordinal will do. Let us thus assume that κ is infinite.

Consider the smallest ordinal ω whose cardinality is strictly greater than that of κ. Then we claim that ω is κ-filtered. Indeed, if C is a collection of at most κ ordinals strictly smaller than ω, then each of these ordinals is of size at most κ. Thus the union of all the ordinals in C (which is an ordinal) is of size at most κ, so is strictly smaller than ω, and it provides an upper bound as in the definition. ▲

Proposition 2.22 *Let M be a module, κ the cardinality of the set of its submodules. Then if ω is κ-filtered, then M is ω-small (with respect to injections).*

The proof is straightforward, but let us first think about a special case. If M is finite, then the claim is that for any inductive system $\{B_\alpha\}$ with injections between them, parametrized by a limit ordinal, any map $M \to \varinjlim B_\alpha$ factors through one of the B_α. But this is clear. M is finite, so since each element in the image must land inside one of the B_α, so all of M lands inside some finite stage.

[4] There are, incidentally, categories, such as the category of rings, where a categorical epimorphism may not be a surjection of sets.

Proof. We need only show that the map Eq. (3.3) is a surjection when ω is κ-filtered. Let $f : A \to \varinjlim B_\alpha$ be a map. Consider the subobjects $\{f^{-1}(B_\alpha)\}$ of A, where B_α is considered as a subobject of the colimit. If one of these, say $f^{-1}(B_\beta)$, fills A, then the map factors through B_β.

So suppose to the contrary that all of the $f^{-1}(B_\alpha)$ were proper subobjects of A. However, we know that

$$\bigcup f^{-1}(B_\alpha) = f^{-1}\left(\bigcup B_\alpha\right) = A.$$

Now there are at most κ different subobjects of A that occur among the $f^{-1}(B_\alpha)$, by hypothesis. Thus we can find a set A of cardinality at most κ such that as α' ranges over A, the $f^{-1}(B_{\alpha'})$ range over *all* the $f^{-1}(B_\alpha)$.

However, A has an upper bound $\widetilde{\omega} < \omega$ as ω is κ-filtered. In particular, all the $f^{-1}(B_{\alpha'})$ are contained in $f^{-1}(B_{\widetilde{\omega}})$. It follows that $f^{-1}(B_{\widetilde{\omega}}) = A$. In particular, the map f factors through $B_{\widetilde{\omega}}$. ▲

From this, we will be able to deduce the existence of lots of injectives. Let us recall the criterion of Baer (**??** 3.16): a module Q is injective if and only if in every commutative diagram

$$\begin{array}{ccc} \mathfrak{a} & \longrightarrow & Q \\ \downarrow & \nearrow & \\ R & & \end{array}$$

for $\mathfrak{a} \subset R$ an ideal, the dotted arrow exists. In other words, we are trying to solve an *extension problem* with respect to the inclusion $\mathfrak{a} \hookrightarrow R$ into the module M.

If M is an R-module, then in general we may have a semi-complete diagram as above. In it, we can form the *push-out*

$$\begin{CD} \mathfrak{a} @>>> Q \\ @VVV @VVV \\ R @>>> R \oplus_{\mathfrak{a}} Q \end{CD} \quad .$$

Here the vertical map is injective, and the diagram commutes. The point is that we can extend $\mathfrak{a} \to Q$ to R *if* we extend Q to the larger module $R \oplus_{\mathfrak{a}} Q$.

The point of the small object argument is to repeat this procedure transfinitely many times.

Theorem 2.23 *Let M be an R-module. Then there is an embedding $M \hookrightarrow Q$ for Q injective.*

Proof. We start by defining a functor $\mathbf{M}$ on the category of R-modules. Given N, we consider the set of all maps $\mathfrak{a} \to N$ for $\mathfrak{a} \subset R$ an ideal, and consider the push-out

$$\begin{CD} \bigoplus \mathfrak{a} @>>> N \\ @VVV @VVV \\ \bigoplus R @>>> N \oplus_{\bigoplus \mathfrak{a}} \bigoplus R \end{CD} \tag{3.4}$$

where the direct sum of copies of R is taken such that every copy of an ideal $\mathfrak{a}$ corresponds to one copy of R. We define $\mathbf{M}(N)$ to be this push-out. Given a map $N \to N'$, there is a natural morphism of diagrams Eq. (3.4), so $\mathbf{M}$ is a functor. Note furthermore that there is a natural transformation

$$N \to \mathbf{M}(N),$$

which is *always an injection.*

The key property of $\mathbf{M}$ is that if $\mathfrak{a} \to N$ is any morphism, it can be extended to $R \to \mathbf{M}(N)$, by the very construction of $\mathbf{M}(N)$. The idea will now be to apply $\mathbf{M}$ a transfinite number of times and to use the small object property.

We define for each ordinal ω a functor $\mathbf{M}_\omega$ on the category of R-modules, together with a natural injection $N \to \mathbf{M}_\omega(N)$. We do this by transfinite induction. First, $\mathbf{M}_1 = \mathbf{M}$ is the functor defined above. Now, suppose given an ordinal ω, and suppose $\mathbf{M}_{\omega'}$ is defined for $\omega' < \omega$. If ω has an immediate predecessor $\widetilde{\omega}$, we let

$$\mathbf{M}_\omega = \mathbf{M} \circ \mathbf{M}_{\widetilde{\omega}}.$$

If not, we let $\mathbf{M}_\omega(N) = \varinjlim_{\omega' < \omega} \mathbf{M}_{\omega'}(N)$. It is clear (e.g. inductively) that the $\mathbf{M}_\omega(N)$ form an inductive system over ordinals ω, so this is reasonable.

Let κ be the cardinality of the set of ideals in R, and let Ω be a κ-filtered ordinal. The claim is as follows.

Lemma 2.24 *For any N, $\mathbf{M}_\Omega(N)$ is injective.*

If we prove this, we will be done. In fact, we will have shown that there is a *functorial* embedding of a module into an injective. Thus, we have only to prove this lemma.

Proof. By Baer's criterion (**??** 3.16), it suffices to show that if $\mathfrak{a} \subset R$ is an ideal, then any map $f : \mathfrak{a} \to \mathbf{M}_\Omega(N)$ extends to $R \to \mathbf{M}_\Omega(N)$. However, we know since Ω is a limit ordinal that

$$\mathbf{M}_\Omega(N) = \varinjlim_{\omega < \Omega} \mathbf{M}_\omega(N),$$

so by Proposition 2.22, we find that

$$\mathrm{Hom}_R(\mathfrak{a}, \mathbf{M}_\Omega(N)) = \varinjlim_{\omega < \Omega} \mathrm{Hom}_R(\mathfrak{a}, \mathbf{M}_\omega(N)).$$

This means in particular that there is some $\omega' < \Omega$ such that f factors through the submodule $\mathbf{M}_{\omega'}(N)$, as

$$f : \mathfrak{a} \to \mathbf{M}_{\omega'}(N) \to \mathbf{M}_\Omega(N).$$

However, by the fundamental property of the functor $\mathbf{M}$, we know that the map $\mathfrak{a} \to \mathbf{M}_{\omega'}(N)$ can be extended to

$$R \to \mathbf{M}(\mathbf{M}_{\omega'}(N)) = \mathbf{M}_{\omega'+1}(N), \qquad \blacktriangle$$

and the last object imbeds in $\mathbf{M}_\Omega(N)$. In particular, f can be extended to $\mathbf{M}_\Omega(N)$. ▲

2.6 Split exact sequences

TO BE ADDED: additive functors preserve split exact seq Suppose that $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{f} N \longrightarrow 0$ is a split short exact sequence. Since $\mathrm{Hom}_R(D, \cdot)$ is a left-exact functor, we see that

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{f'} \mathrm{Hom}_R(D, N)$$

is exact. In addition, $\mathrm{Hom}_R(D, L \oplus N) \cong \mathrm{Hom}_R(D, L) \oplus \mathrm{Hom}_R(D, N)$. Therefore, in the case that we start with a split short exact sequence $M \cong L \oplus N$, applying $\mathrm{Hom}_R(D, \cdot)$ does yield a split short exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{f'} \mathrm{Hom}_R(D, N) \longrightarrow 0.$$

Now, assume that

$$0 \longrightarrow \mathrm{Hom}_R(D,L) \xrightarrow{\psi'} \mathrm{Hom}_R(D,M) \xrightarrow{f'} \mathrm{Hom}_R(D,N) \longrightarrow 0$$

is a short exact sequence of abelian groups for all R-modules D. Set $D = R$ and using $\mathrm{Hom}_R(R,N) \cong N$ yields that $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{f} N \longrightarrow 0$ is a short exact sequence.

Set $D = N$, so we have

$$0 \longrightarrow \mathrm{Hom}_R(N,L) \xrightarrow{\psi'} \mathrm{Hom}_R(N,M) \xrightarrow{f'} \mathrm{Hom}_R(N,N) \longrightarrow 0$$

Here, f' is surjective, so the identity map of $\mathrm{Hom}_R(N,N)$ lifts to a map $g \in \mathrm{Hom}_R(N,M)$ so that $f \circ g = f'(g) = id$. This means that g is a splitting homomorphism for the sequence $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{f} N \longrightarrow 0$, and therefore the sequence is a split short exact sequence.

§3 The tensor product

We shall now introduce the third functor of this chapter: the tensor product. The tensor product's key property is that it allows one to "linearize" bilinear maps. When taking the tensor product of rings, it provides a categorical coproduct as well.

3.1 Bilinear maps and the tensor product

Let R be a commutative ring, as usual. We have seen that the Hom-sets $\mathrm{Hom}_R(M,N)$ of R-modules M, N are themselves R-modules. Consequently, if we have three R-modules M, N, P, we can think about module-homomorphisms

$$M \stackrel{\lambda}{\to} \mathrm{Hom}_R(N,P).$$

Suppose $x \in M, y \in N$. Then we can consider $\lambda(x) \in \mathrm{Hom}_R(N,P)$ and thus we can consider the element $\lambda(x)(y) \in P$. We denote this element $\lambda(x)(y)$, which depends on the variables $x \in M, y \in N$, by $\lambda(x,y)$ for convenience; it is a function of two variables $M \times N \to P$.

There are certain properties of $\lambda(\cdot,\cdot)$ that we list below. Fix $x, x' \in M$; $y, y' \in N$; $a \in R$. Then:

1. $\lambda(x, y+y') = \lambda(x,y) + \lambda(x,y')$ because $\lambda(x)$ is additive.
2. $\lambda(x, ay) = a\lambda(x,y)$ because $\lambda(x)$ is an R-module homomorphism.
3. $\lambda(x+x', y) = \lambda(x,y) + \lambda(x',y)$ because λ is additive.
4. $\lambda(ax, y) = a\lambda(x,y)$ because λ is an R-module homomorphism.

Conversely, given a function $\lambda : M \times N \to P$ of two variables satisfying the above properties, it is easy to see that we can get a morphism of R-modules $M \to \mathrm{Hom}_R(N,P)$.

Definition 3.1 An ***R*-bilinear map** $\lambda : M \times N \to P$ is a map satisfying the above listed conditions. In other words, it is required to be R-linear in each variable separately.

The previous discussion shows that there is a *bijection* between R-bilinear maps $M \times N \to P$ with R-module maps $M \to \mathrm{Hom}_R(N,P)$. Note that the first interpretation is symmetric in M, N; the second, by contrast, can be interpreted in terms of the old concepts of an R-module map. So both are useful.

Exercise 3.17 Prove that a $\mathbb{Z}$-bilinear map out of $\mathbb{Z}/2 \times \mathbb{Z}/3$ is identically zero, whatever the target module.

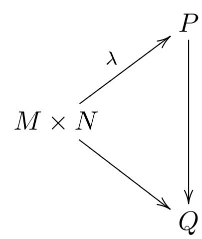
Let us keep the notation of the previous discussion: in particular, M, N, P will be modules over a commutative ring R.

Given a bilinear map $M \times N \to P$ and a homomorphism $P \to P'$, we can clearly get a bilinear map $M \times N \to P'$ by composition. In particular, given M, N, there is a *covariant functor* from R-modules to **Sets** sending any R-module P to the collection of R-bilinear maps $M \times N \to P$. As usual, we are interested in when this functor is *corepresentable.* As a result, we are interested in *universal* bilinear maps out of $M \times N$.

Definition 3.2 An R-bilinear map $\lambda : M \times N \to P$ is called **universal** if for all R-modules Q, the composition of $P \to Q$ with $M \times N \stackrel{\lambda}{\to} P$ gives a **bijection**

$$\mathrm{Hom}_R(P, Q) \simeq \{\text{bilinear maps } M \times N \to Q\}$$

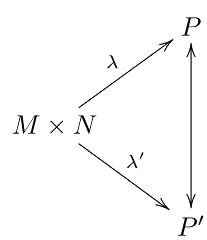
So, given a bilinear map $M \times N \to Q$, there is a *unique* map $P \to Q$ making the diagram



Alternatively, P *corepresents* the functor $Q \to \{\text{bilinear maps } M \times N \to Q\}$.

General nonsense says that given M, N, an universal R-bilinear map $M \times N \to P$ is **unique** up to isomorphism (if it exists). This follows from *Yoneda's lemma.* For convenience, we give a direct proof.

Suppose $M \times N \stackrel{\lambda}{\to} P$ was universal and $M \times N \stackrel{\lambda'}{\to} P'$ is also universal. Then by the universal property, there are unique maps $P \to P'$ and $P' \to P$ making the following diagram commutative:



These compositions $P \to P' \to P, P' \to P \to P'$ have to be the identity because of the uniqueness part of the universal property. As a result, $P \to P'$ is an isomorphism.

We shall now show that this universal object does indeed exist.

Proposition 3.3 *Given M, N, a universal bilinear map out of $M \times N$ exists.*

Before proving it we make:

Definition 3.4 We denote the codomain of the universal map out of $M \times N$ by $M \otimes_R N$. This is called the **tensor product** of M, N, so there is a universal bilinear map out of $M \times N$ into $M \otimes_R N$.

Proof (Proof of Proposition 3.3). We will simply give a presentation of the tensor product by "generators and relations." Take the free R-module $M \otimes_R N$ generated by the symbols $\{x \otimes y\}_{x\in M, y\in N}$ and quotient out by the relations forced upon us by the definition of a bilinear map (for $x, x' \in M,\ y, y' \in N,\ a \in R$)

1. $(x + x') \otimes y = x \otimes y + x' \otimes y$.
2. $(ax) \otimes y = a(x \otimes y) = x \otimes (ay)$.
3. $x \otimes (y + y') = x \otimes y + x \otimes y'$.

We will abuse notation and denote $x \otimes y$ for its image in $M \otimes_R N$ (as opposed to the symbol generating the free module).

There is a bilinear map $M \times N \to M \otimes_R N$ sending $(x, y) \to x \otimes y$; the relations imposed imply that this map is a bilinear map. We have to check that it is universal, but this is actually quite direct.

Suppose we had a bilinear map $\lambda : M \times N \to P$. We must construct a linear map $M \otimes_R N \to P$. To do this, we can just give a map on generators, and show that it is zero on each of the relations. It is easy to see that to make the appropriate diagrams commute, the linear map $M \otimes N \to P$ has to send $x \otimes y \to \lambda(x, y)$. This factors through the relations on $x \otimes y$ by bilinearity and leads to an R-linear map $M \otimes_R N \to P$ such that the following diagram commutes:

$$\begin{array}{ccc} M \times N & \longrightarrow & M \otimes_R N \\ & \searrow^{\lambda} & \downarrow \\ & & P \end{array}.$$

It is easy to see that $M \otimes_R N \to P$ is unique because the $x \otimes y$ generate it. ▲

The theory of the tensor product allows one to do away with bilinear maps and just think of linear maps.

Given M, N, we have constructed an object $M \otimes_R N$. We now wish to see the functoriality of the tensor product. In fact, $(M, N) \to M \otimes_R N$ is a *covariant functor* in two variables from R-modules to R-modules. In particular, if $M \to M', N \to N'$ are morphisms, there is a canonical map

$$M \otimes_R N \to M' \otimes_R N'. \tag{3.5}$$

To obtain Eq. (3.5), we take the natural bilinear map $M \times N \to M' \times N' \to M' \otimes_R N'$ and use the universal property of $M \otimes_R N$ to get a map out of it.

3.2 Basic properties of the tensor product

We make some observations and prove a few basic properties. As the proofs will show, one powerful way to prove things about an object is to reason about its universal property. If two objects have the same universal property, they are isomorphic.

Proposition 3.5 *The tensor product is symmetric: for R-modules M, N, we have $M \otimes_R N \simeq N \otimes_R M$ canonically.*

Proof. This is clear from the universal properties: giving a bilinear map out of $M \times N$ is the same as a bilinear map out $N \times M$. Thus $M \otimes_R N$ and $N \otimes_R N$ have the same universal property. It is also clear from the explicit construction. ▲

Proposition 3.6 *For an R-module M, there is a canonical isomorphism $M \to M \otimes_R R$.*

Proof. If we think in terms of bilinear maps, this statement is equivalent to the statement that a bilinear map $\lambda : M \times R \to P$ is the same as a linear map $M \to N$. Indeed, to do this, restrict λ to $\lambda(\cdot, 1)$. Given $f : M \to N$, similarly, we take for λ as $\lambda(x, a) = af(x)$. This gives a bijection as claimed. ▲

Proposition 3.7 *The tensor product is associative. There are canonical isomorphisms $M \otimes_R (N \otimes_R P) \simeq (M \otimes_R N) \otimes_R P$.*

Proof. There are a few ways to see this: one is to build it explicitly from the construction given, sending $x \otimes (y \otimes z) \to (x \otimes y) \otimes z$.

More conceptually, both have the same universal property: by general categorical nonsense (Yoneda's lemma), we need to show that for all Q, there is a canonical bijection

$$\mathrm{Hom}_R(M \otimes (N \otimes P)), Q) \simeq \mathrm{Hom}_R((M \otimes N) \otimes P, Q)$$

where the R's are dropped for simplicity. But both of these sets can be identified with the set of trilinear maps[5] $M \times N \times P \to Q$. Indeed

$$\begin{aligned} \mathrm{Hom}_R(M \otimes (N \otimes P), Q) &\simeq \text{bilinear } M \times (N \otimes P) \to Q \\ &\simeq \mathrm{Hom}(N \otimes P, \mathrm{Hom}(M, Q)) \\ &\simeq \text{bilinear } N \times P \to \mathrm{Hom}(M, Q) \\ &\simeq \mathrm{Hom}(N, \mathrm{Hom}(P, \mathrm{Hom}(M, Q)) \\ &\simeq \text{trilinear maps.} \end{aligned}$$ ▲

3.3 The adjoint property

Finally, while we defined the tensor product in terms of a "universal bilinear map," we saw earlier that bilinear maps could be interpreted as maps into a suitable Hom-set. In particular, fix R-modules M, N, P. We know that the set of bilinear maps $M \times N \to P$ is naturally in bijection with

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$$

as well as with

$$\mathrm{Hom}_R(M \otimes_R, N, P).$$

As a result, we find:

Proposition 3.8 *For R-modules M, N, P, there is a natural bijection*

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \simeq \mathrm{Hom}_R(M \otimes_R N, P).$$

There is a more evocative way of phrasing the above natural bijection. Given N, let us define the functors F_N, G_N via

$$F_N(M) = M \otimes_R N, \quad G_N(P) = \mathrm{Hom}_R(N, P).$$

Then the above proposition states that there is a natural isomorphism

$$\mathrm{Hom}_R(F_N(M), P) \simeq \mathrm{Hom}_R(M, G_N(P)).$$

In particular, F_N and G_N are *adjoint functors*. So, in a sense, the operations of Hom and $\otimes$ are dual to each other.

[5]Easy to define.

Proposition 3.9 *Tensoring commutes with colimits.*

In particular, it follows that if $\{N_\alpha\}$ is a family of modules, and M is a module, then

$$M \otimes_R \bigoplus N_\alpha = \bigoplus M \otimes_R N_\alpha.$$

EXERCISE 3.18 Give an explicit proof of the above relation.

Proof. This is a formal consequence of the fact that the tensor product is a left adjoint and consequently commutes with all colimits. **TO BE ADDED:** proof ▲

In particular, by Proposition 3.9, the tensor product commutes with *cokernels.* That is, if $A \to B \to C \to 0$ is an exact sequence of R-modules and M is an R-module, $A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is also exact, because exactness of such a sequence is precisely a condition on the cokernel. That is, the tensor product is *right exact.*

We can thus prove a simple result on finite generation:

Proposition 3.10 *If M, N are finitely generated, then $M \otimes_R N$ is finitely generated.*

Proof. Indeed, if we have surjections $R^m \to M, R^n \to N$, we can tensor them; we get a surjection since the tensor product is right-exact. So have a surjection $R^{mn} = R^m \otimes_R R^n \to M \otimes_R N$. ▲

3.4 The tensor product as base-change

Before this, we have considered the tensor product as a functor within a fixed category. Now, we shall see that when one takes the tensor product with a *ring*, one gets additional structure. As a result, we will be able to get natural functors between *different* module categories.

Suppose we have a ring-homomorphism $\phi : R \to R'$. In this case, any R'-module can be regarded as an R-module. In particular, there is a canonical functor of *restriction*

$$R'\text{-modules} \to R\text{-modules}.$$

We shall see that the tensor product provides an *adjoint* to this functor. Namely, if M has an R-module structure, then $M \otimes_R R'$ has an R' module structure where R' acts on the right. Since the tensor product is functorial, this gives a functor in the opposite direction:

$$R\text{-modules} \to R'\text{-modules}.$$

Let M' be an R'-module and M an R-module. In view of the above, we can talk about

$$\mathrm{Hom}_R(M, M')$$

by thinking of M' as an R-module.

Proposition 3.11 *There is a canonical isomorphism between*

$$\mathrm{Hom}_R(M, M') \simeq \mathrm{Hom}_{R'}(M \otimes_R R', M').$$

In particular, the restriction functor and the functor $M \to M \otimes_R R'$ are adjoints to each other.

Proof. We can describe the bijection explicitly. Given an R'-homomorphism $f : M \otimes_R R' \to M'$, we get a map

$$f_0 : M \to M'$$

sending

$$m \to m \otimes 1 \to f(m \otimes 1).$$

This is easily seen to be an R-module-homomorphism. Indeed,

$$f_0(ax) = f(ax \otimes 1) = f(\phi(a)(x \otimes 1)) = af(x \otimes 1) = af_0(x)$$

since f is an R'-module homomorphism.

Conversely, if we are given a homomorphism of R-modules

$$f_0 : M \to M'$$

then we can define

$$f : M \otimes_R R' \to M'$$

by sending $m \otimes r' \to r' f_0(m)$, which is a homomorphism of R' modules. This is well-defined because f_0 is a homomorphism of R-modules. We leave some details to the reader. ▲

Example 3.12 In the representation theory of finite groups, the operation of tensor product corresponds to the procedure of *inducing* a representation. Namely, if $H \subset G$ is a subgroup of a group G, then there is an obvious restriction functor from G-representations to H-representations. The adjoint to this is the induction operator. Since a H-representation (resp. a G-representation) is just a module over the group ring, the operation of induction is really a special case of the tensor product. Note that the group rings are generally not commutative, so this should be interpreted with some care.

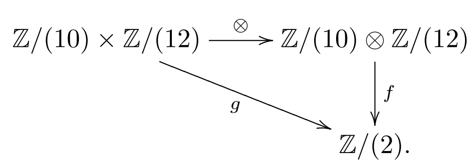
3.5 Some concrete examples

We now present several concrete computations of tensor products in explicit cases to illuminate what is happening.

Example 3.13 Let us compute $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/12$. Since 1 spans $\mathbb{Z}/(10)$ and 1 spans $\mathbb{Z}/(12)$, we see that $1 \otimes 1$ spans $\mathbb{Z}/(10) \otimes \mathbb{Z}/(12)$ and this tensor product is a cyclic group.

Note that $1 \otimes 0 = 1 \otimes (10 \cdot 0) = 10 \otimes 0 = 0 \otimes 0 = 0$ and $0 \otimes 1 = (12 \cdot 0) \otimes 1 = 0 \otimes 12 = 0 \otimes 0 = 0$. Now, $10(1 \otimes 1) = 10 \otimes 1 = 0 \otimes 1 = 0$ and $12(1 \otimes 1) = 1 \otimes 12 = 1 \otimes 0 = 0$, so the cyclic group $\mathbb{Z}/(10) \otimes \mathbb{Z}/(12)$ has order dividing both 10 and 12. This means that the cyclic group has order dividing $\gcd(10, 12) = 2$.

To show that the order of $\mathbb{Z}/(10) \otimes \mathbb{Z}/(12)$, define a bilinear map $g : \mathbb{Z}/(10) \times \mathbb{Z}/(12) \to \mathbb{Z}/(2)$ via $g : (x, y) \mapsto xy$. The universal property of tensor products then says that there is a unique linear map $f : \mathbb{Z}/(10) \otimes \mathbb{Z}/(12) \to \mathbb{Z}/(2)$ making the diagram



commute. In particular, this means that $f(x \otimes y) = g(x, y) = xy$. Hence, $f(1 \otimes 1) = 1$, so f is surjective, and therefore, $\mathbb{Z}/(10) \otimes \mathbb{Z}/(12)$ has size at least two. This allows us to conclude that $\mathbb{Z}/(10) \otimes \mathbb{Z}/(12) = \mathbb{Z}/(2)$.

We now generalize the above example to tensor products of cyclic groups.

Example 3.14 Let $d = \gcd(m, n)$. We will show that $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/d\mathbb{Z})$, and thus in particular if m and n are relatively prime, then $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) \simeq (0)$. First, note that any $a \otimes b \in (\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$ can be written as $ab(1 \otimes 1)$, so that $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$ is generated by $1 \otimes 1$ and hence is a cyclic group. We know from elementary number theory that $d = xm + yn$ for some $x, y \in \mathbb{Z}$. We have $m(1 \otimes 1) = m \otimes 1 = 0 \otimes 1 = 0$ and $n(1 \otimes 1) = 1 \otimes n = 1 \otimes 0 = 0$. Thus $d(1 \otimes 1) = (xm + yn)(1 \otimes 1) = 0$, so that $1 \otimes 1$ has order dividing d.

Conversely, consider the map $f : (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/d\mathbb{Z})$ defined by $f(a + m\mathbb{Z}, b + n\mathbb{Z}) = ab + d\mathbb{Z}$. This is well-defined, since if $a' + m\mathbb{Z} = a + m\mathbb{Z}$ and $b' + n\mathbb{Z} = b + n\mathbb{Z}$ then $a' = a + mr$ and $b' = b + ns$ for some r, s and thus $a'b' + d\mathbb{Z} = ab + (mrb + nsa + mnrs) + d\mathbb{Z} = ab + d\mathbb{Z}$ (since $d = \gcd(m, n)$ divides m and n). This is obviously bilinear, and hence induces a map $\tilde{f} : (\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/d\mathbb{Z})$, which has $\tilde{f}(1 \otimes 1) = 1 + d\mathbb{Z}$. But the order of $1 + d\mathbb{Z}$ in $\mathbb{Z}/d\mathbb{Z}$ is d, so that the order of $1 \otimes 1$ in $(\mathbb{Z}/m\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$ must be at least d. Thus $1 \otimes 1$ is in fact of order d, and the map $\tilde{f}$ is an isomorphism between cyclic groups of order d.

Finally, we present an example involving the interaction of Hom and the tensor product.

Example 3.15 Given an R-module M, let us use the notation $M^* = \mathrm{Hom}_R(M, R)$. We shall define a functorial map

$$M^* \otimes_R N \to \mathrm{Hom}_R(M, N),$$

and show that it is an isomorphism when M is finitely generated and free.

Define $\rho' : M^* \times N \to \mathrm{Hom}_R(M, N)$ by $\rho'(f, n)(m) = f(m)n$ (note that $f(m) \in R$, and the multiplication $f(m)n$ is that between an element of R and an element of N). This is bilinear,

$$\rho'(af+bg, n)(m) = (af+bg)(m)n = (af(m)+bg(m))n = af(m)n+bg(m)n = a\rho'(f, n)(m)+b\rho'(g, n)(m)$$

$$\rho'(f, an_1 + bn_2)(m) = f(m)(an_1 + bn_2) = af(m)n_1 + bf(m)n_2 = a\rho'(f, n_1)(m) + b\rho'(f, n_2)(m)$$

so it induces a map $\rho : M^* \otimes N \to \mathrm{Hom}(M, N)$ with $\rho(f \otimes n)(m) = f(m)n$. This homomorphism is unique since the $f \otimes n$ generate $M^* \otimes N$.

Suppose M is free on the set $\{a_1, \ldots, a_k\}$. Then $M^* = \mathrm{Hom}(M, R)$ is free on the set $\{f_i : M \to R, f_i(r_1 a_1 + \cdots + r_k a_k) = r_i\}$, because there are clearly no relations among the f_i and because any $f : M \to R$ has $f = f(a_1)f_1 + \cdots + f(a_n)f_n$. Also note that any element $\sum h_j \otimes p_j \in M^* \otimes N$ can be written in the form $\sum_{i=1}^k f_i \otimes n_i$, by setting $n_i = \sum h_j(a_i)p_j$, and *that this is unique* because the f_i are a basis for M^*.

We claim that the map $\psi : \mathrm{Hom}_R(M, N) \to M^* \otimes N$ defined by $\psi(g) = \sum_{i=1}^k f_i \otimes g(a_i)$ is inverse to ρ. Given any $\sum_{i=1}^k f_i \otimes n_i \in M^* \otimes N$, we have

$$\rho(\sum_{i=1}^k f_i \otimes n_i)(a_j) = \sum_{i=1}^k \rho(f_i \otimes n_i)(a_j) = \sum_{i=1}^k f_i(a_j)n_i = n_j$$

Thus, $\rho(\sum_{i=1}^k f_i \otimes n_i)(a_i) = n_i$, and thus $\psi(\rho(\sum_{i=1}^k f_i \otimes n_i)) = \sum_{i=1}^k f_i \otimes n_i$. Thus, $\psi \circ \rho = \mathrm{id}_{M^* \otimes N}$.

Conversely, recall that for $g : M \to N \in \mathrm{Hom}_R(M, N)$, we defined $\psi(g) = \sum_{i=1}^k f_i \otimes g(a_i)$. Thus,

$$\rho(\psi(g))(a_j) = \rho(\sum_{i=1}^k f_i \otimes g(a_i))(a_j) = \sum_{i=1}^k \rho(f_i \otimes g(a_i))(a_j) = \sum_{i=1}^k f_i(a_j)g(a_i) = g(a_j)$$

and because $\rho(\psi(g))$ agrees with g on the a_i, it is the same element of $\mathrm{Hom}_R(M, N)$ because the a_i generate M. Thus, $\rho \circ \psi = \mathrm{id}_{\mathrm{Hom}_R(M,N)}$.

Thus, ρ is an isomorphism.

We now interpret localization as a tensor product.

Proposition 3.16 *Let R be a commutative ring, $S \subset R$ a multiplicative subset. Then there exists a canonical isomorphism of functors:*

$$\phi : S^{-1}M \simeq S^{-1}R \otimes_R M.$$

Proof. Here is a construction of ϕ. If $x/s \in S^{-1}M$ where $x \in M, s \in S$, we define

$$\phi(x/s) = (1/s) \otimes m.$$

Let us check that this is well-defined. Suppose $x/s = x'/s'$; then this means there is $t \in S$ with

$$xs't = x'st.$$

From this we need to check that $\phi(x/s) = \phi(x'/s')$, i.e. that $1/s \otimes x$ and $1/s' \otimes x'$ represent the same elements in the tensor product. But we know from the last statement that

$$\frac{1}{ss't} \otimes xs't = \frac{1}{ss't} x'st \in S^{-1}R \otimes M$$

and the first is just

$$s't(\frac{1}{ss't} \otimes x) = \frac{1}{s} \otimes x$$

by linearity, while the second is just

$$\frac{1}{s'} \otimes x'$$

similarly. One next checks that ϕ is an R-module homomorphism, which we leave to the reader.

Finally, we need to describe the inverse. The inverse $\psi : S^{-1}R \otimes M \to S^{-1}M$ is easy to construct because it's a map out of the tensor product, and we just need to give a bilinear map

$$S^{-1}R \times M \to S^{-1}M,$$

and this sends $(r/s, m)$ to mr/s.

It is easy to see that ϕ, ψ are inverses to each other by the definitions. ▲

It is, perhaps, worth making a small categorical comment, and offering an alternative argument. We are given two functors F, G from R-modules to $S^{-1}R$-modules, where $F(M) = S^{-1}R \otimes_R M$ and $G(M) = S^{-1}M$. By the universal property, the map $M \to S^{-1}M$ from an R-module to a tensor product gives a natural map

$$S^{-1}R \otimes_R M \to S^{-1}M,$$

that is a natural transformation $F \to G$. Since it is an isomorphism for free modules, it is an isomorphism for all modules by a standard argument.

3.6 Tensor products of algebras

There is one other basic property of tensor products to discuss before moving on: namely, what happens when one tensors a ring with another ring. We shall see that this gives rise to *push-outs* in the category of rings, or alternatively, coproducts in the category of R-algebras. Let R be a commutative ring and suppose R_1, R_2 are R-algebras. That is, we have ring homomorphisms $\phi_0 : R \to R_0, \quad \phi_1 : R \to R_1$.

Proposition 3.17 *$R_0 \otimes_R R_1$ has the structure of a commutative ring in a natural way.*

Indeed, this multiplication multiplies two typical elements $x \otimes y, x' \otimes y'$ of the tensor product by sending them to $xx' \otimes yy'$. The ring structure is determined by this formula. One ought to check that this approach respects the relations of the tensor product. We will do so in an indirect way.

Proof. Notice that giving a multiplication law on $R_0 \otimes_R R_1$ is equivalent to giving an R-bilinear map

$$(R_0 \otimes_R R_1) \times (R_0 \otimes R_1) \to R_0 \otimes_R R_1,$$

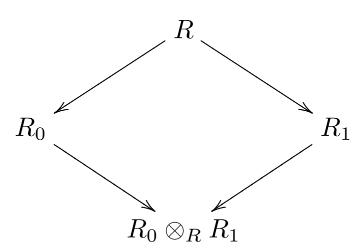
i.e. an R-linear map

$$(R_0 \otimes_R R_1) \otimes_R (R_0 \otimes R_1) \to R_0 \otimes_R R_1$$

which satisfies certain constraints (associativity, commutativity, etc.). But the left side is isomorphic to $(R_0 \otimes_R R_0) \otimes_R (R_1 \otimes_R R_1)$. Since we have bilinear maps $R_0 \times R_0 \to R_0$ and $R_1 \times R_1 \to R_1$, we get linear maps $R_0 \otimes_R R_0 \to R_0$ and $R_1 \otimes_R R_1 \to R_1$. Tensoring these maps gives the multiplication as a bilinear map. It is easy to see that these two approaches are the same.

We now need to check that this operation is commutative and associative, with $1 \otimes 1$ as a unit; moreover, it distributes over addition. Distributivity over addition is built into the construction (i.e. in view of bilinearity). The rest (commutativity, associativity, units) can be checked directly on the generators, since we have distributivity. We shall leave the details to the reader. ▲

We can in fact describe the tensor product of R-algebras by a universal property. We will describe a commutative diagram:



Here $R_0 \to R_0 \otimes_R R_1$ sends $x \mapsto x \otimes 1$; similarly for $R_1 \mapsto R_0 \otimes_R R_1$. These are ring-homomorphisms, and it is easy to see that the above diagram commutes, since $r \otimes 1 = 1 \otimes r = r(1 \otimes 1)$ for $r \in R$. In fact,

Proposition 3.18 *$R_0 \otimes_R R_1$ is universal with respect to this property: in the language of category theory, the above diagram is a pushout square.*

This means for any commutative ring B, and every pair of maps $u_0 : R_0 \to B$ and $u_1 : R_1 \to B$ such that the pull-backs $R \to R_0 \to B$ and $R \to R_1 \to B$ are the same, then we get a unique map of rings

$$R_0 \otimes_R R_1 \to B$$

which restricts on R_0, R_1 to the morphisms u_0, u_1 that we started with.

Proof. If B is a ring as in the previous paragraph, we make B into an R-module by the map $R \to R_0 \to B$ (or $R \to R_1 \to B$, it is the same by assumption). This map $R_0 \otimes_R R_1 \to B$ sends

$$x \otimes y \to u_0(x)u_1(y).$$

It is easy to check that $(x, y) \to u_0(x)u_1(y)$ is R-bilinear (because of the condition that the two pull-backs of u_0, u_1 to R are the same), and that it gives a homomorphism of rings $R_0 \otimes_R R_1 \to B$ which restricts to u_0, u_1 on R_0, R_1. One can check, for instance, that this is a homomorphism of rings by looking at the generators.

It is also clear that $R_0 \otimes_R R_1 \to B$ is unique, because we know that the map on elements of the form $x \otimes 1$ and $1 \otimes y$ is determined by u_0, u_1; these generate $R_0 \otimes_R R_1$, though. ▲

In fact, we now claim that the category of rings has *all* coproducts. We see that the coproduct of any two elements exists (as the tensor product over $\mathbb{Z}$). It turns out that arbitrary coproducts exist. More generally, if $\{R_\alpha\}$ is a family of R-algebras, then one can define an object

$$\bigotimes_\alpha R_\alpha,$$

which is a coproduct of the R_α in the category of R-algebras. To do this, we simply take the generators as before, as formal objects

$$\bigotimes r_\alpha, \quad r_\alpha \in R_\alpha,$$

except that all but finitely many of the r_α are required to be the identity. One quotients by the usual relations.

Alternatively, one may use the fact that filtered colimits exist, and construct the infinite coproduct as a colimit of finite coproducts (which are just ordinary tensor products).

§4 Exactness properties of the tensor product

In general, the tensor product is not exact; it is only exact on the right, but it can fail to preserve injections. Yet in some important cases it *is* exact. We study that in the present section.

4.1 Right-exactness of the tensor product

We will start by talking about extent to which tensor products do preserve exactness under any circumstance. First, let's recall what is going on. If M, N are R-modules over the commutative ring R, we have defined another R-module $\mathrm{Hom}_R(M, N)$ of morphisms $M \to N$. This is left-exact as a functor of N. In other words, if we fix M and let N vary, then the construction of homming out of M preserves kernels.

In the language of category theory, this construction $N \to \mathrm{Hom}_R(M, N)$ has an adjoint. The other construction we discussed last time was this adjoint, and it is the tensor product. Namely, given M, N we defined a **tensor product** $M \otimes_R N$ such that giving a map $M \otimes_R N \to P$ into some R-module P is the same as giving a bilinear map $\lambda : M \times N \to P$, which in turn is the same as giving an R-linear map

$$M \to \mathrm{Hom}_R(N, P).$$

So we have a functorial isomorphism

$$\mathrm{Hom}_R(M \otimes_R N, P) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)).$$

Alternatively, tensoring is the left-adjoint to the hom functor. By abstract nonsense, it follows that since $\mathrm{Hom}(M, \cdot)$ preserves cokernels, the left-adjoint preserves cokernels and is right-exact. We shall see this directly.

Proposition 4.1 *The functor $N \to M \otimes_R N$ is right-exact, i.e. preserves cokernels.*

In fact, the tensor product is symmetric, so it's right exact in either variable.

Proof. We have to show that if $N' \to N \to N'' \to 0$ is exact, then so is

$$M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0.$$

There are a lot of different ways to think about this. For instance, we can look at the direct construction. The tensor product is a certain quotient of a free module.

$M \otimes_R N''$ is the quotient of the free module generated by $m \otimes n'', m \in M, n \in N''$ modulo the usual relations. The map $M \otimes N \to M \otimes N''$ sends $m \otimes n \to m \otimes n''$ if n'' is the image of n in N''. Since each n'' can be lifted to some n, it is obvious that the map $M \otimes_R N \to M \otimes_R N''$ is surjective.

Now we know that $M \otimes_R N''$ is a quotient of $M \otimes_R N$. But which relations do you have to impose on $M \otimes_R N$ to get $M \otimes_R N''$? In fact, each relation in $M \otimes_R N''$ can be lifted to a relation in $M \otimes_R N$, but with some redundancy. So the only thing to quotient out by is the statement that $x \otimes y, x \otimes y'$ have the same image in $M \otimes N''$. In particular, we have to quotient out by

$$x \otimes y - x \otimes y' \ , y - y' \in N'$$

so that if we kill off $x \otimes n'$ for $n' \in N' \subset N$, then we get $M \otimes N''$. This is a direct proof.

One can also give a conceptual proof. We would like to know that $M \otimes N''$ is the cokernel of $M \otimes N' \to M \otimes N''$. In other words, we'd like to know that if we mapped $M \otimes_R N$ into some P and the pull-back to $M \otimes_R N'$, it'd factor uniquely through $M \otimes_R N''$. Namely, we need to show that

$$\mathrm{Hom}_R(M \otimes N'', P) = \ker(\mathrm{Hom}_R(M \otimes N, P) \to \mathrm{Hom}_R(M \otimes N'', P)).$$

But the first is just $\mathrm{Hom}_R(N'', \mathrm{Hom}_R(M, P))$ by the adjointness property. Similarly, the second is just

$$\ker(\mathrm{Hom}_R(N, \mathrm{Hom}(M, P)) \to \mathrm{Hom}_R(N', \mathrm{Hom}_R(M, P))$$

but this last statement is $\mathrm{Hom}_R(N'', \mathrm{Hom}_R(M, P))$ by just the statement that $N'' = \mathrm{coker}(N' \to N)$. To give a map N'' into some module (e.g. $\mathrm{Hom}_R(M, P)$) is the same thing as giving a map out of N which kills N'. So we get the functorial isomorphism. ▲

Remark Formation of tensor products is, in general, **not** exact.

Example 4.2 Let $R = \mathbb{Z}$. Let $M = \mathbb{Z}/2\mathbb{Z}$. Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

which we can tensor with M, yielding

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \to 0$$

I claim that the second thing $\mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z}$ is zero. This is because by tensoring with $\mathbb{Z}/2\mathbb{Z}$, we've made multiplication by 2 identically zero. By tensoring with $\mathbb{Q}$, we've made multiplication by 2 invertible. The only way to reconcile this is to have the second term zero. In particular, the sequence becomes

$$0 \to \mathbb{Z}/2\mathbb{Z} \to 0 \to 0 \to 0$$

which is not exact.

Exercise 3.19 Let R be a ring, $I, J \subset R$ ideals. Show that $R/I \otimes_R R/J \simeq R/(I + J)$.

4.2 A characterization of right-exact functors

Let us consider additive functors on the category of R-modules. So far, we know a very easy way of getting such functors: given an R-module N, we have a functor

$$T_N : M \to M \otimes_R N.$$

In other words, we have a way of generating a functor on the category of R-modules for each R-module. These functors are all right-exact, as we have seen. Now we will prove a converse.

Proposition 4.3 *Let F be a right-exact functor on the category of R-modules that commutes with direct sums. Then F is isomorphic to some T_N.*

Proof. The idea is that N will be $F(R)$.

Without the right-exactness hypothesis, we shall construct a natural morphism

$$F(R) \otimes M \to F(M)$$

as follows. Given $m \in M$, there is a natural map $R \to M$ sending $1 \to m$. This identifies $M = \mathrm{Hom}_R(R, M)$. But functoriality gives a map $F(R) \times \mathrm{Hom}_R(R, M) \to F(M)$, which is clearly R-linear; the universal property of the tensor product now produces the desired transformation $T_{F(R)} \to F$.

It is clear that $T_{F(R)}(M) \to F(M)$ is an isomorphism for $M = R$, and thus for M free, as both $T_{F(R)}$ and F commute with direct sums. Now let M be any R-module. There is a "free presentation," that is an exact sequence

$$R^I \to R^J \to M \to 0$$

for some sets I, J; we get a commutative, exact diagram

$$\begin{array}{ccccccc}
T_{F(R)}(R^I) & \longrightarrow & T_{F(R)}(R^J) & \longrightarrow & T_{F(R)}(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
F(R^I) & \longrightarrow & F(R^J) & \longrightarrow & F(M) & \longrightarrow & 0
\end{array}$$

where the leftmost two vertical arrows are isomorphisms. A diagram chase now shows that $T_{F(R)}(M) \to F(M)$ is an isomorphism. In particular, $F \simeq T_{F(R)}$ as functors. ▲

Without the hypothesis that F commutes with arbitrary direct sums, we could only draw the same conclusion on the category of *finitely presented* modules; the same proof as above goes through, though I and J are required to be finite.[6]

Proposition 4.4 *Let F be a right-exact functor on the category of finitely presented R-modules that commutes with direct sums. Then F is isomorphic to some T_N.*

From this we can easily see that localization at a multiplicative subset $S \subset R$ is given by tensoring with $S^{-1}R$. Indeed, localization is a right-exact functor on the category of R-modules, so it is given by tensoring with some module M; applying to R shows that $M = S^{-1}R$.

[6]Recall that an additive functor commutes with finite direct sums.

4.3 Flatness

In some cases, though, the tensor product is exact.

Definition 4.5 Let R be a commutative ring. An R-module M is called **flat** if the functor $N \to M \otimes_R N$ is exact. An R-algebra is **flat** if it is flat as an R-module.

We already know that tensoring with anything is right exact, so the only thing to be checked for flatness of M is that the operation of tensoring by M preserves injections.

Example 4.6 $\mathbb{Z}/2\mathbb{Z}$ is not flat as a $\mathbb{Z}$-module by Example 4.2.

Example 4.7 If R is a ring, then R is flat as an R-module, because tensoring by R is the identity functor.

More generally, if P is a projective module (i.e., homming out of P is exact), then P is flat.

Proof. If $P = \bigoplus_A R$ is free, then tensoring with P corresponds to taking the direct sum $|A|$ times, i.e.

$$P \otimes_R M = \bigoplus_A M.$$

This is because tensoring with R preserves (finite or direct) infinite sums. The functor $M \to \bigoplus_A M$ is exact, so free modules are flat.

A projective module, as discussed earlier, is a direct summand of a free module. So if P is projective, $P \oplus P' \simeq \bigoplus_A R$ for some P'. Then we have that

$$(P \otimes_R M) \oplus (P' \otimes_R M) \simeq \bigoplus_A M.$$

If we had an injection $M \to M'$, then there is a direct sum decomposition yields a diagram of maps

$$\begin{array}{ccc} P \otimes_R M & \longrightarrow & \bigoplus_A M \\ \downarrow & & \downarrow \\ P \otimes_R M' & \longrightarrow & \bigoplus_A M' \end{array}.$$

A diagram-chase now shows that the vertical map is injective. Namely, the composition $P \otimes_R M \to \bigoplus_A M'$ is injective, so the vertical map has to be injective too. ▲

Example 4.8 If $S \subset R$ is a multiplicative subset, then $S^{-1}R$ is a flat R-module, because localization is an exact functor.

Let us make a few other comments.

Remark Let $\phi : R \to R'$ be a homomorphism of rings. Then, first of all, any R'-module can be regarded as an R-module by composition with ϕ. In particular, R' is an R-module.

If M is an R-module, we can define

$$M \otimes_R R'$$

as an R-module. But in fact this tensor product is an R'-module; it has an action of R'. If $x \in M$ and $a \in R'$ and $b \in R'$, multiplication of $(x \otimes a) \in M \otimes_R R'$ by $b \in R'$ sends this, *by definition*, to

$$b(x \otimes a) = x \otimes ab.$$

It is easy to check that this defines an action of R' on $M \otimes_R R'$. (One has to check that this action factors through the appropriate relations, etc.)

The following fact shows that the hom-sets behave nicely with respect to flat base change.

Proposition 4.9 *Let M be a finitely presented R-module, N an R-module. Let S be a flat R-algebra. Then the natural map*

$$\mathrm{Hom}_R(M, N) \otimes_R S \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$$

is an isomorphism.

Proof. Indeed, it is clear that there is a natural map

$$\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$$

of R-modules. The latter is an S-module, so the universal property gives the map $\mathrm{Hom}_R(M, N)\otimes_R S \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$ as claimed. If N is fixed, then we have two contravariant functors in M,

$$T_1(M) = \mathrm{Hom}_R(M, N) \otimes_R S, \quad T_2(M) = \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S).$$

We also have a natural transformation $T_1(M) \to T_2(M)$. It is clear that if M is *finitely generated* and *free*, then the natural transformation is an isomorphism (for example, if $M = R$, then we just have the map $N \otimes_R S \to N \otimes_R S$).

Note moreover that both functors are left-exact: that is, given an exact sequence

$$M' \to M \to M'' \to 0,$$

there are induced exact sequences

$$0 \to T_1(M'') \to T_1(M) \to T_1(M'), \quad 0 \to T_2(M'') \to T_2(M) \to T_2(M').$$

Here we are using the fact that Hom is always a left-exact functor and the fact that tensoring with S preserves exactness. (Thus it is here that we use flatness.)

Now the following lemma will complete the proof:

Lemma 4.10 *Let T_1, T_2 be contravariant, left-exact additive functors from the category of R-modules to the category of abelian groups. Suppose a natural transformation $t : T_1(M) \to T_2(M)$ is given, and suppose this is an isomorphism whenever M is finitely generated and free. Then it is an isomorphism for any finitely presented module M.*

Proof. This lemma is a diagram chase. Fix a finitely presented M, and choose a presentation

$$F' \to F \to M \to 0,$$

with F', F finitely generated and free. Then we have an exact and commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_1(M) & \longrightarrow & T_1(F) & \longrightarrow & T_1(F') \\ & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & T_2(M) & \longrightarrow & T_2(F) & \longrightarrow & T_2(F'). \end{array}$$

▲

By hypotheses, the two vertical arrows to the right are isomorphisms, as indicated. A diagram chase now shows that the remaining arrow is an isomorphism, which is what we wanted to prove. ▲

Example 4.11 Let us now consider finitely generated flat modules over a principal ideal domain R. By Theorem 5.4, we know that any such M is isomorphic to a direct sum $\bigoplus R/a_i$ for some $a_i \in R$. But if any of the a_i is not zero, then that a_i would be a nonzero zerodivisor on M. However, we know no element of $R - \{0\}$ can be a zerodivisor on M. It follows that all the $a_i = 0$. In particular, we have proved:

Proposition 4.12 *A finitely generated module over a PID is flat if and only if it is free.*

4.4 Finitely presented flat modules

In Example 4.7, we saw that a projective module over any ring R was automatically flat. In general, the converse is flat. For instance, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module (as tensoring by $\mathbb{Q}$ is a form of localization). However, because $\mathbb{Q}$ is divisible (namely, multiplication by n is surjective for any n), $\mathbb{Q}$ cannot be a free abelian group.

Nonetheless:

Theorem 4.13 *A finitely presented flat module over a ring R is projective.*

Proof. We follow [Wei94].

Let us define the following contravariant functor from R-modules to R-modules. Given M, we send it to $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. This is made into an R-module in the following manner: given $\phi : M \to \mathbb{Q}/\mathbb{Z}$ (which is just a homomorphism of abelian groups!) and $r \in R$, we send this to $r\phi$ defined by $(r\phi)(m) = \phi(rm)$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group, we see that $M \mapsto M^*$ is an *exact* contravariant functor from R-modules to R-modules. In fact, we note that $0 \to A \to B \to C \to 0$ is exact implies $0 \to C^* \to B^* \to A^* \to 0$ is exact.

Let F be any R-module. There is a natural homomorphism

$$M^* \otimes_R F \to \mathrm{Hom}_R(F, M)^*. \tag{3.6}$$

This is defined as follows. Given $\phi : M \to \mathbb{Q}/\mathbb{Z}$ and $x \in F$, we define a new map $\mathrm{Hom}(F, M) \to \mathbb{Q}/\mathbb{Z}$ by sending a homomorphism $\psi : F \to M$ to $\phi(\psi(x))$. In other words, we have a natural map

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \otimes_R F \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_R(F, M)^*, \mathbb{Q}/\mathbb{Z}).$$

Now fix M. This map (3.6) is an isomorphism if F is *finitely generated* and free. Both are right-exact (because dualizing is contravariant-exact!). The "finite presentation trick" now shows that the map is an isomorphism if F is finitely presented. **TO BE ADDED:** this should be elaborated on

Fix now F finitely presented and flat, and consider the above two quantities in (3.6) as functors in M. Then the first functor is exact, so the second one is too. In particular, $\mathrm{Hom}_R(F, M)^*$ is an exact functor in M; in particular, if $M \twoheadrightarrow M''$ is a surjection, then

$$\mathrm{Hom}_R(F, M'')^* \to \mathrm{Hom}_R(F, M)^*$$

is an injection. But this implies that

$$\mathrm{Hom}_R(F, M) \to \mathrm{Hom}_R(F, M'')$$

is a *surjection,* i.e. that F is projective. Indeed:

Lemma 4.14 *$A \to B \to C$ is exact if and only if $C^* \to B^* \to A^*$ is exact.*

Proof. Indeed, one direction was already clear (from $\mathbb{Q}/\mathbb{Z}$ being an injective abelian group). Conversely, we note that $M = 0$ if and only if $M^* = 0$ (again by injectivity and the fact that $(\mathbb{Z}/a)^* \neq 0$ for any a). Thus dualizing reflects isomorphisms: if a map becomes an isomorphism after dualized, then it was an isomorphism already. From here it is easy to deduce the result (by applying the above fact to the kernel and image). ▲

CRing Project contents

CRing Project bibliography

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[Bou98] Nicolas Bourbaki. *Commutative algebra. Chapters 1–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[Cam88] Oscar Campoli. A principal ideal domain that is not a euclidean domain. *American Mathematical Monthly*, 95(9):868–871, 1988.

[CF86] J. W. S. Cassels and A. Fröhlich, editors. *Algebraic number theory*, London, 1986. Academic Press Inc. [Harcourt Brace Jovanovich Publishers]. Reprint of the 1967 original.

[Cla11] Pete L. Clark. Factorization in euclidean domains. 2011. Available at `http://math.uga.edu/~pete/factorization2010.pdf`.

[dJea10] Aise Johan de Jong et al. *Stacks Project*. Open source project, available at `http://www.math.columbia.edu/algebraic_geometry/stacks-git/`, 2010.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[For91] Otto Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Translated from the 1977 German original by Bruce Gilligan, Reprint of the 1981 English translation.

[GD] Alexander Grothendieck and Jean Dieudonné. *Élements de géometrie algébrique*. Publications Mathématiques de l'IHÉS.

[Ger] Anton Geraschenko (mathoverflow.net/users/1). Is there an example of a formally smooth morphism which is not smooth? MathOverflow. `http://mathoverflow.net/questions/200` (version: 2009-10-08).

[Gil70] Robert Gilmer. An existence theorem for non-Noetherian rings. *The American Mathematical Monthly*, 77(6):621–623, 1970.

[Gre97] John Greene. Principal ideal domains are almost euclidean. *The American Mathematical Monthly*, 104(2):154–156, 1997.

[Gro57] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002. Available at `http://www.math.cornell.edu/~hatcher/AT/AT.pdf`.

[Hov07] Mark Hovey. *Model Categories.* American Mathematical Society, 2007.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 2006.

[Lan94] Serge Lang. *Algebraic number theory*, volume 110 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1994.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, third edition, 2002.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics.* Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[LR08] T. Y. Lam and Manuel L. Reyes. A prime ideal principle in commutative algebra. *J. Algebra*, 319(7):3006–3027, 2008.

[Mar02] David Marker. *Model theory*, volume 217 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2002. An introduction.

[Mat80] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series.* Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.

[McC76] John McCabe. A note on Zariski's lemma. *The American Mathematical Monthly*, 83(7):560–561, 1976.

[Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series.* Princeton University Press, Princeton, N.J., 1980.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[Per04] Hervé Perdry. An elementary proof of Krull's intersection theorem. *The American Mathematical Monthly*, 111(4):356–357, 2004.

[Qui] Daniel Quillen. Homology of commutative rings. Mimeographed notes.

[Ray70] Michel Raynaud. *Anneaux locaux henséliens.* Lecture Notes in Mathematics, Vol. 169. Springer-Verlag, Berlin, 1970.

[RG71] Michel Raynaud and Laurent Gruson. Critères de platitude et de projectivité. Techniques de "platification" d'un module. *Invent. Math.*, 13:1–89, 1971.

[Ser65] Jean-Pierre Serre. *Algèbre locale. Multiplicités*, volume 11 of *Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel. Seconde édition, 1965. Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1965.

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

[Ser09] Jean-Pierre Serre. How to use finite fields for problems concerning infinite fields. 2009. arXiv:0903.0517v2.

[SGA72] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[SGA03] *Revêtements étales et groupe fondamental (SGA 1).* Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)].

[Tam94] Günter Tamme. *Introduction to étale cohomology.* Universitext. Springer-Verlag, Berlin, 1994. Translated from the German by Manfred Kolster.

[Vis08] Angelo Vistoli. Notes on Grothendieck topologies, fibered categories, and descent theory. *Published in* FGA Explained, 2008. arXiv:math/0412512v4.

[Was97] Lawrence C. Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1997.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994.