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Chapter 4

The Spec of a ring

The notion of the Spec of a ring is fundamental in modern algebraic geometry. It is the scheme-theoretic analog of classical affine schemes. The identification occurs when one identifies the maximal ideals of the polynomial ring $k[x_1, \dots, x_n]$ (for k an algebraically closed field) with the points of the classical variety $\mathbb{A}_k^n = k^n$. In modern algebraic geometry, one adds the “non-closed points” given by the other prime ideals. Just as general varieties were classically defined by gluing affine varieties, a scheme is defined by gluing open affines.

This is not a book on schemes, but it will nonetheless be convenient to introduce the Spec construction, outside of the obvious benefits of including preparatory material for algebraic geometry. First of all, it will provide a convenient notation. Second, and more importantly, it will provide a convenient geometric intuition. For example, an R -module can be thought of as a kind of “vector bundle”—technically, a sheaf—over the space $\text{Spec } R$, with the caveat that the rank might not be locally constant (which is, however, the case when the module is projective).

§1 The spectrum of a ring

We shall now associate to every commutative ring a topological space $\text{Spec } R$ in a functorial manner. That is, there will be a contravariant functor

$$\text{Spec} : \mathbf{CRing} \rightarrow \mathbf{Top}$$

where \mathbf{Top} is the category of topological spaces. This construction is the basis for scheme-theoretic algebraic geometry and will be used frequently in the sequel.

The motivating observation is the following. If k is an algebraically closed field, then the maximal ideals in $k[x_1, \dots, x_n]$ are of the form $(x_1 - a_1, \dots, x_n - a_n)$ for $(a_1, \dots, a_n) \in k[x_1, \dots, x_n]$. This is the Nullstellensatz, which we have not proved yet. We can thus identify the maximal ideals in the polynomial ring with the space k^n . If $I \subset k[x_1, \dots, x_n]$ is an ideal, then the maximal ideals in $k[x_1, \dots, x_n]$ correspond to points where everything in I vanishes. See Example 1.5 for a more detailed explanation. Classical affine algebraic geometry thus studies the set of maximal ideals in an algebra finitely generated over an algebraically closed field.

The Spec of a ring is a generalization of this construction. In general, it is more natural to use all prime ideals instead of just maximal ideals.

1.1 Definition and examples

We start by defining Spec as a set. We will next construct the Zariski topology and later the functoriality.

Definition 1.1 Let R be a commutative ring. The **spectrum** of R , denoted $\text{Spec } R$, is the set of prime ideals of R .

We shall now make $\text{Spec } R$ into a topological space. First, we describe a collection of sets which will become the closed sets. If $I \subset R$ is an ideal, let

$$V(I) = \{\mathfrak{p} : \mathfrak{p} \supset I\} \subset \text{Spec } R.$$

Proposition 1.2 *There is a topology on $\text{Spec } R$ such that the closed subsets are of the form $V(I)$ for $I \subset R$ an ideal.*

Proof. Indeed, we have to check the familiar axioms for a topology:

1. $\emptyset = V((1))$ because no prime contains 1. So \emptyset is closed.
2. $\text{Spec } R = V((0))$ because any ideal contains zero. So $\text{Spec } R$ is closed.
3. We show the closed sets are stable under intersections. Let $K_\alpha = V(I_\alpha)$ be closed subsets of $\text{Spec } R$ for α ranging over some index set. Let $I = \sum I_\alpha$. Then

$$V(I) = \bigcap K_\alpha = \bigcap V(I_\alpha),$$

which follows because I is the smallest ideal containing each I_α , so a prime contains every I_α iff it contains I .

4. The union of two closed sets is closed. Indeed, if $K, K' \subset \text{Spec } R$ are closed, we show $K \cup K'$ is closed. Say $K = V(I), K' = V(I')$. Then we claim:

$$K \cup K' = V(II').$$

Here, as usual, II' is the ideal generated by products $ii', i \in I, i' \in I'$. If \mathfrak{p} is **prime** and contains II' , it must contain one of I, I' ; this implies the displayed equation above and implies the result. ▲

Definition 1.3 The topology on $\text{Spec } R$ defined above is called the **Zariski topology**. With it, $\text{Spec } R$ is now a topological space.

EXERCISE 4.1 What is the Spec of the zero ring?

In order to see the geometry of this construction, let us work several examples.

Example 1.4 Let $R = \mathbb{Z}$, and consider $\text{Spec } \mathbb{Z}$. Then every prime is generated by one element, since \mathbb{Z} is a PID. We have that $\text{Spec } \mathbb{Z} = \{(0)\} \cup \bigcup_{p \text{ prime}} \{(p)\}$. The picture is that one has all the familiar primes $(2), (3), (5), \dots$, and then a special point (0) .

Let us now describe the closed subsets. These are of the form $V(I)$ where $I \subset \mathbb{Z}$ is an ideal, so $I = (n)$ for some $n \in \mathbb{Z}$.

1. If $n = 0$, the closed subset is all of $\text{Spec } \mathbb{Z}$.
2. If $n \neq 0$, then n has finitely many prime divisors. So $V((n))$ consists of the prime ideals corresponding to these prime divisors.

The only closed subsets besides the entire space are the finite subsets that exclude (0) .

Example 1.5 Say $R = \mathbb{C}[x, y]$ is a polynomial ring in two variables. We will not give a complete description of $\text{Spec } R$ here. But we will write down several prime ideals.

1. For every pair of complex numbers $s, t \in \mathbb{C}$, the collection of polynomials $f \in R$ such that $f(s, t) = 0$ is a prime ideal $\mathfrak{m}_{s,t} \subset R$. In fact, it is maximal, as the residue ring is all of \mathbb{C} . Indeed, $R/\mathfrak{m}_{s,t} \simeq \mathbb{C}$ under the map $f \rightarrow f(s, t)$.

In fact,

Theorem 1.6 *The $\mathfrak{m}_{s,t}$ are all the maximal ideals in R .*

This will follow from the *Hilbert Nullstellensatz* to be proved later (Theorem 4.5).

2. $(0) \subset R$ is a prime ideal since R is a domain.
3. If $f(x, y) \in R$ is an irreducible polynomial, then (f) is a prime ideal. This is equivalent to unique factorization in R .¹

To draw $\text{Spec } R$, we start by drawing \mathbb{C}^2 , which is identified with the collection of maximal ideals $\mathfrak{m}_{s,t}$, $s, t \in \mathbb{C}$. $\text{Spec } R$ has additional (non-closed) points too, as described above, but for now let us consider the topology induced on \mathbb{C}^2 as a subspace of $\text{Spec } R$.

The closed subsets of $\text{Spec } R$ are subsets $V(I)$ where I is an ideal, generated by polynomials $\{f_\alpha(x, y)\}$. It is of interest to determine the subset of \mathbb{C}^2 that $V(I)$ induces. In other words, we ask:

What points of \mathbb{C}^2 (with (s, t) identified with $\mathfrak{m}_{s,t}$) lie in $V(I)$?

Now, by definition, we know that (s, t) corresponds to a point of $V(I)$ if and only if $I \subset \mathfrak{m}_{s,t}$. This is true iff all the f_α lie in $\mathfrak{m}_{s,t}$, i.e. if $f_\alpha(s, t) = 0$ for all α . So the closed subsets of \mathbb{C}^2 (with the induced Zariski topology) are *precisely the subsets that can be defined by polynomial equations*.

This is **much** coarser than the usual topology. For instance, $\{(z_1, z_2) : \Re(z_1) \geq 0\}$ is not Zariski-closed. The Zariski topology is so coarse because one has only algebraic data (namely, polynomials, or elements of R) to define the topology.

EXERCISE 4.2 Let R_1, R_2 be commutative rings. Give $R_1 \times R_2$ a natural structure of a ring, and describe $\text{Spec}(R_1 \times R_2)$ in terms of $\text{Spec } R_1$ and $\text{Spec } R_2$.

EXERCISE 4.3 Let X be a compact Hausdorff space, $C(X)$ the ring of real continuous functions $X \rightarrow \mathbb{R}$. The maximal ideals in $\text{Spec } C(X)$ are in bijection with the points of X , and the topology induced on X (as a subset of $\text{Spec } C(X)$ with the Zariski topology) is just the usual topology.

EXERCISE 4.4 Prove the following result: if X, Y are compact Hausdorff spaces and $C(X), C(Y)$ the associated rings of continuous functions, if $C(X), C(Y)$ are isomorphic as \mathbb{R} -algebras, then X is homeomorphic to Y .

1.2 The radical ideal-closed subset correspondence

We now return to the case of an arbitrary commutative ring R . If $I \subset R$, we get a closed subset $V(I) \subset \text{Spec } R$. It is called $V(I)$ because one is supposed to think of it as the places where the elements of I “vanish,” as the elements of R are something like “functions.” This analogy is perhaps best seen in the example of a polynomial ring over an algebraically closed field, e.g. Example 1.5 above.

The map from ideals into closed sets is very far from being injective in general, though by definition it is surjective.

Example 1.7 If $R = \mathbb{Z}$ and p is prime, then $I = (p), I' = (p^2)$ define the same subset (namely, $\{(p)\}$) of $\text{Spec } R$.

¹To be proved later ??.

We now ask why the map from ideals to closed subsets fails to be injective. As we shall see, the entire problem disappears if we restrict to *radical* ideals.

Definition 1.8 If I is an ideal, then the **radical** $\text{Rad}(I)$ or \sqrt{I} is defined as

$$\text{Rad}(I) = \{x \in R : x^n \in I \text{ for some } n\}.$$

An ideal is **radical** if it is equal to its radical. (This is equivalent to the earlier Definition 2.5.)

Before proceeding, we must check:

Lemma 1.9 *If I an ideal, so is $\text{Rad}(I)$.*

Proof. Clearly $\text{Rad}(I)$ is closed under multiplication since I is. Suppose $x, y \in \text{Rad}(I)$; we show $x + y \in \text{Rad}(I)$. Then $x^n, y^n \in I$ for some n (large) and thus for all larger n . The binomial expansion now gives

$$(x + y)^{2n} = x^{2n} + \binom{2n}{1} x^{2n-1} y + \cdots + y^{2n},$$

where every term contains either x, y with power $\geq n$, so every term belongs to I . Thus $(x+y)^{2n} \in I$ and, by definition, we see then that $x + y \in \text{Rad}(I)$. \blacktriangle

The map $I \rightarrow V(I)$ does in fact depend only on the radical of I . In fact, if I, J have the same radical $\text{Rad}(I) = \text{Rad}(J)$, then $V(I) = V(J)$. Indeed, $V(I) = V(\text{Rad}(I)) = V(\text{Rad}(J)) = V(J)$ by:

Lemma 1.10 *For any I , $V(I) = V(\text{Rad}(I))$.*

Proof. Indeed, $I \subset \text{Rad}(I)$ and therefore obviously $V(\text{Rad}(I)) \subset V(I)$. We have to show the converse inclusion. Namely, we must prove:

If $\mathfrak{p} \supset I$, then $\mathfrak{p} \supset \text{Rad}(I)$.

So suppose $\mathfrak{p} \supset I$ is prime and $x \in \text{Rad}(I)$; then $x^n \in I \subset \mathfrak{p}$ for some n . But \mathfrak{p} is prime, so whenever a product of things belongs to \mathfrak{p} , a factor does. Thus since $x^n = x \cdot x \cdots x$, we must have $x \in \mathfrak{p}$. So

$$\text{Rad}(I) \subset \mathfrak{p},$$

proving the quoted claim, and thus the lemma. \blacktriangle

There is a converse to this remark:

Proposition 1.11 *If $V(I) = V(J)$, then $\text{Rad}(I) = \text{Rad}(J)$.*

So two ideals define the same closed subset iff they have the same radical.

Proof. We write down a formula for $\text{Rad}(I)$ that will imply this at once.

Lemma 1.12 *For a commutative ring R and an ideal $I \subset R$,*

$$\text{Rad}(I) = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}.$$

From this, it follows that $V(I)$ determines $\text{Rad}(I)$. This will thus imply the proposition. We now prove the lemma:

Proof. 1. We show $\text{Rad}(I) \subset \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$. In particular, this follows if we show that if a prime contains I , it contains $\text{Rad}(I)$; but we have already discussed this above.

2. If $x \notin \text{Rad}(I)$, we will show that there is a prime ideal $\mathfrak{p} \supset I$ not containing x . This will imply the reverse inclusion and the lemma.

We want to find \mathfrak{p} not containing x , more generally not containing any power of x . In particular, we want $\mathfrak{p} \cap \{1, x, x^2, \dots\} = \emptyset$. This set $S = \{1, x, \dots\}$ is multiplicatively closed, in that it contains 1 and is closed under finite products. Right now, it does not intersect I ; we want to find a *prime* containing I that still does not intersect $\{x^n, n \geq 0\}$.

More generally, we will prove:

Sublemma 1.13 *Let S be multiplicatively closed set in any ring R and let I be any ideal with $I \cap S = \emptyset$. There is a prime ideal $\mathfrak{p} \supset I$ and does not intersect S (in fact, any ideal maximal with respect to the condition of not intersecting S will do).*

In English, any ideal missing S can be enlarged to a prime ideal missing S . This is actually fancier version of a previous argument. We showed earlier that any ideal not containing the multiplicatively closed subset $\{1\}$ can be contained in a prime ideal not containing 1, in Proposition 4.5.

Note that the sublemma clearly implies the lemma when applied to $S = \{1, x, \dots\}$.

Proof (Proof of the sublemma). Let $P = \{J : J \supset I, J \cap S = \emptyset\}$. Then P is a poset with respect to inclusion. Note that $P \neq \emptyset$ because $I \in P$. Also, for any nonempty linearly ordered subset of P , the union is in P (i.e. there is an upper bound). We can invoke Zorn's lemma to get a maximal element of P . This element is an ideal $\mathfrak{p} \supset I$ with $\mathfrak{p} \cap S = \emptyset$. We claim that \mathfrak{p} is prime.

First of all, $1 \notin \mathfrak{p}$ because $1 \in S$. We need only check that if $xy \in \mathfrak{p}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Suppose otherwise, so $x, y \notin \mathfrak{p}$. Then $(x, \mathfrak{p}) \notin P$ or \mathfrak{p} would not be maximal. Ditto for (y, \mathfrak{p}) .

In particular, we have that these bigger ideals both intersect S . This means that there are

$$a \in \mathfrak{p}, r \in R \quad \text{such that} \quad a + rx \in S$$

and

$$b \in \mathfrak{p}, r' \in R \quad \text{such that} \quad b + r'y \in S.$$

Now S is multiplicatively closed, so multiply $(a + rx)(b + r'y) \in S$. We find:

$$ab + ar'y + brx + rr'xy \in S. \quad \blacktriangle$$

Now $a, b \in \mathfrak{p}$ and $xy \in \mathfrak{p}$, so all the terms above are in \mathfrak{p} , and the sum is too. But this contradicts $\mathfrak{p} \cap S = \emptyset$. \blacktriangle

The upshot of the previous lemmata is:

Proposition 1.14 *There is a bijection between the closed subsets of $\text{Spec } R$ and radical ideals $I \subset R$.*

1.3 A meta-observation about prime ideals

We saw in the previous subsection (?? 1.13) that an ideal maximal with respect to the property of not intersecting a multiplicatively closed subset is prime. It turns out that this is the case for many such properties of ideals. A general method of seeing this was developed in [LR08]. In this (optional) subsection, we digress to explain this phenomenon.

If I is an ideal and $a \in R$, we define the notation

$$(I : a) = \{x \in R : xa \in I\}.$$

More generally, if J is an ideal, we define

$$(I : J) = \{x \in R : xJ \subset I\}.$$

Let R be a ring, and \mathcal{F} a collection of ideals of R . We are interested in conditions that will guarantee that the maximal elements of \mathcal{F} are *prime*. Actually, we will do the opposite: the following condition will guarantee that the ideals maximal at *not* being in \mathcal{F} are prime.

Definition 1.15 The family \mathcal{F} is called an **Oka family** if $R \in \mathcal{F}$ (where R is considered as an ideal) and whenever $I \subset R$ is an ideal and $(I : a), (I, a) \in \mathcal{F}$ (for some $a \in R$), then $I \in \mathcal{F}$.

Example 1.16 Let us begin with a simple observation. If $(I : a)$ is generated by a_1, \dots, a_n and (I, a) is generated by a, b_1, \dots, b_m (where we may take $b_1, \dots, b_m \in I$, without loss of generality), then I is generated by $aa_1, \dots, aa_n, b_1, \dots, b_m$. To see this, note that if $x \in I$, then $x \in (I, a)$ is a linear combination of the $\{a, b_1, \dots, b_m\}$, but the coefficient of a must lie in $(I : a)$.

As a result, we may deduce that the family of finitely generated ideals is an Oka family.

Example 1.17 Let us now show that the family of *principal* ideals is an Oka family. Indeed, suppose $I \subset R$ is an ideal, and (I, a) and $(I : a)$ are principal. One can easily check that $(I : a) = (I : (I, a))$. Setting $J = (I, a)$, we find that J is principal and $(I : J)$ is too. However, for *any* principal ideal J , and for any ideal $I \subset J$,

$$I = J(I : J)$$

as one easily checks. Thus we find in our situation that since $J = (I, a)$ and $(I : J)$ are principal, I is principal.

Proposition 1.18 ([LR08]) *If \mathcal{F} is an Oka family of ideals, then any maximal element of the complement of \mathcal{F} is prime.*

Proof. Suppose $I \notin \mathcal{F}$ is maximal with respect to not being in \mathcal{F} but I is not prime. Note that $I \neq R$ by hypothesis. Then there is $a \in R$ such that $(I : a), (I, a)$ both strictly contain I , so they must belong to \mathcal{F} . Indeed, we can find $a, b \in R - I$ with $ab \in I$; it follows that $(I, a) \neq I$ and $(I : a)$ contains $b \notin I$.

By the Oka condition, we have $I \in \mathcal{F}$, a contradiction. ▲

Corollary 1.19 (Cohen) *If every prime ideal of R is finitely generated, then every ideal of R is finitely generated.*²

Proof. Suppose that there existed ideals $I \subset R$ which were not finitely generated. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that are not finitely generated is not finitely generated; indeed, if $I = \bigcup I_\alpha$ were generated by a_1, \dots, a_n , then all the generators would belong to some I_α and would consequently generate it.

By Zorn's lemma, there is an ideal maximal with respect to being not finitely generated. However, by Proposition 1.18, this ideal is necessarily prime (since the family of finitely generated ideals is an Oka family). This contradicts the hypothesis. ▲

Corollary 1.20 *If every prime ideal of R is principal, then every ideal of R is principal.*

Proof. This is proved in the same way. ▲

EXERCISE 4.5 Suppose every nonzero prime ideal in R contains a non-zero-divisor. Then R is a domain. (Hint: consider the set S of nonzerodivisors, and argue that any ideal maximal with respect to not intersecting S is prime. Thus, (0) is prime.)

²Later we will say that R is *noetherian*.

Remark Let R be a ring. Let κ be an infinite cardinal. By applying Example 1.16 and Proposition 1.18 we see that any ideal maximal with respect to the property of not being generated by κ elements is prime. This result is not so useful because there exists a ring for which every prime ideal of R can be generated by \aleph_0 elements, but some ideal cannot. Namely, let k be a field, let T be a set whose cardinality is greater than \aleph_0 and let

$$R = k[\{x_n\}_{n \geq 1}, \{z_{t,n}\}_{t \in T, n \geq 0}] / (x_n^2, z_{t,n}^2, x_n z_{t,n} - z_{t,n-1})$$

This is a local ring with unique prime ideal $\mathfrak{m} = (x_n)$. But the ideal $(z_{t,n})$ cannot be generated by countably many elements.

1.4 Functoriality of Spec

The construction $R \rightarrow \text{Spec } R$ is functorial in R in a contravariant sense. That is, if $f : R \rightarrow R'$, there is a continuous map $\text{Spec } R' \rightarrow \text{Spec } R$. This map sends $\mathfrak{p} \subset R'$ to $f^{-1}(\mathfrak{p}) \subset R$, which is easily seen to be a prime ideal in R . Call this map $F : \text{Spec } R' \rightarrow \text{Spec } R$. So far, we have seen that $\text{Spec } R$ induces a contravariant functor from **Rings** \rightarrow **Sets**.

EXERCISE 4.6 A contravariant functor $F : \mathcal{C} \rightarrow \mathbf{Sets}$ (for some category \mathcal{C}) is called **representable** if it is naturally isomorphic to a functor of the form $X \rightarrow \text{Hom}(X, X_0)$ for some $X_0 \in \mathcal{C}$, or equivalently if the induced covariant functor on \mathcal{C}^{op} is corepresentable.

The functor $R \rightarrow \text{Spec } R$ is not representable. (Hint: Indeed, a representable functor must send the initial object into a one-point set.)

Next, we check that the morphisms induced on Spec 's from a ring-homomorphism are in fact *continuous* maps of topological spaces.

Proposition 1.21 *Spec induces a contravariant functor from **Rings** to the category **Top** of topological spaces.*

Proof. Let $f : R \rightarrow R'$. We need to check that this map $\text{Spec } R' \rightarrow \text{Spec } R$, which we call F , is continuous. That is, we must check that F^{-1} sends closed subsets of $\text{Spec } R$ to closed subsets of $\text{Spec } R'$.

More precisely, if $I \subset R$ and we take the inverse image $F^{-1}(V(I)) \subset \text{Spec } R'$, it is just the closed set $V(f(I))$. This is best left to the reader, but here is the justification. If $\mathfrak{p} \in \text{Spec } R'$, then $F(\mathfrak{p}) = f^{-1}(\mathfrak{p}) \supset I$ if and only if $\mathfrak{p} \supset f(I)$. So $F(\mathfrak{p}) \in V(I)$ if and only if $\mathfrak{p} \in V(f(I))$.

Example 1.22 Let R be a commutative ring, $I \subset R$ an ideal, $f : R \rightarrow R/I$. There is a map of topological spaces

$$F : \text{Spec}(R/I) \rightarrow \text{Spec } R.$$

This map is a closed embedding whose image is $V(I)$. Most of this follows because there is a bijection between ideals of R containing I and ideals of R/I , and this bijection preserves primality.

EXERCISE 4.7 Show that this map $\text{Spec } R/I \rightarrow \text{Spec } R$ is indeed a homeomorphism from $\text{Spec } R/I \rightarrow V(I)$.

1.5 A basis for the Zariski topology

In the previous section, we were talking about the Zariski topology. If R is a commutative ring, we recall that $\text{Spec } R$ is defined to be the collection of prime ideals in R . This has a topology where the closed sets are the sets of the form

$$V(I) = \{\mathfrak{p} \in \text{Spec } R : \mathfrak{p} \supset I\}.$$

There is another way to describe the Zariski topology in terms of *open* sets.

Definition 1.23 If $f \in R$, we let

$$U_f = \{\mathfrak{p} : f \notin \mathfrak{p}\}$$

so that U_f is the subset of $\text{Spec } R$ consisting of primes not containing f . This is the complement of $V((f))$, so it is open.

Proposition 1.24 *The sets U_f form a basis for the Zariski topology.*

Proof. Suppose $U \subset \text{Spec } R$ is open. We claim that U is a union of basic open sets U_f .

Now $U = \text{Spec } R - V(I)$ for some ideal I . Then

$$U = \bigcup_{f \in I} U_f$$

because if an ideal is not in $V(I)$, then it fails to contain some $f \in I$, i.e. is in U_f for that f . Alternatively, we could take complements, whence the above statement becomes

$$V(I) = \bigcap_{f \in I} V((f))$$

which is clear. ▲

The basic open sets have nice properties.

1. $U_1 = \text{Spec } R$ because prime ideals are not allowed to contain the unit element.
2. $U_0 = \emptyset$ because every prime ideal contains 0.
3. $U_{fg} = U_f \cap U_g$ because fg lies in a prime ideal \mathfrak{p} if and only if one of f, g does.

Now let us describe what the Zariski topology has to do with localization. Let R be a ring and $f \in R$. Consider $S = \{1, f, f^2, \dots\}$; this is a multiplicatively closed subset. Last week, we defined $S^{-1}R$.

Definition 1.25 For S the powers of f , we write R_f or $R[f^{-1}]$ for the localization $S^{-1}R$.

There is a map $\phi : R \rightarrow R[f^{-1}]$ and a corresponding map

$$\text{Spec } R[f^{-1}] \rightarrow \text{Spec } R$$

sending a prime $\mathfrak{p} \subset R[f^{-1}]$ to $\phi^{-1}(\mathfrak{p})$.

Proposition 1.26 *This map induces a homeomorphism of $\text{Spec } R[f^{-1}]$ onto $U_f \subset \text{Spec } R$.*

So if one takes a commutative ring and inverts an element, one just gets an open subset of Spec . This is why it's called localization: one is restricting to an open subset on the Spec level when one inverts something.

Proof. The reader is encouraged to work this proof out for herself.

1. First, we show that $\text{Spec } R[f^{-1}] \rightarrow \text{Spec } R$ lands in U_f . If $\mathfrak{p} \subset R[f^{-1}]$, then we must show that the inverse image $\phi^{-1}(\mathfrak{p})$ can't contain f . If otherwise, that would imply that $\phi(f) \in \mathfrak{p}$; however, $\phi(f)$ is invertible, and then \mathfrak{p} would be (1).

2. Let's show that the map surjects onto U_f . If $\mathfrak{p} \subset R$ is a prime ideal not containing f , i.e. $\mathfrak{p} \in U_f$. We want to construct a corresponding prime in the ring $R[f^{-1}]$ whose inverse image is \mathfrak{p} .

Let $\mathfrak{p}[f^{-1}]$ be the collection of all fractions

$$\left\{ \frac{x}{f^n}, x \in \mathfrak{p} \right\} \subset R[f^{-1}],$$

which is evidently an ideal. Note that whether the numerator is in \mathfrak{p} is **independent** of the representing fraction $\frac{x}{f^n}$ used.³ In fact, $\mathfrak{p}[f^{-1}]$ is a prime ideal. Indeed, suppose

$$\frac{a}{f^m} \frac{b}{f^n} \in \mathfrak{p}[f^{-1}].$$

Then $\frac{ab}{f^{m+n}}$ belongs to this ideal, which means $ab \in \mathfrak{p}$; so one of $a, b \in \mathfrak{p}$ and one of the two fractions $\frac{a}{f^m}, \frac{b}{f^n}$ belongs to $\mathfrak{p}[f^{-1}]$. Also, $1/1 \notin \mathfrak{p}[f^{-1}]$.

It is clear that the inverse image of $\mathfrak{p}[f^{-1}]$ is \mathfrak{p} , because the image of $x \in R$ is $x/1$, and this belongs to $\mathfrak{p}[f^{-1}]$ precisely when $x \in \mathfrak{p}$.

3. The map $\text{Spec } R[f^{-1}] \rightarrow \text{Spec } R$ is injective. Suppose $\mathfrak{p}, \mathfrak{p}'$ are prime ideals in the localization and the inverse images are the same. We must show that $\mathfrak{p} = \mathfrak{p}'$.

Suppose $\frac{x}{f^n} \in \mathfrak{p}$. Then $x/1 \in \mathfrak{p}$, so $x \in \phi^{-1}(\mathfrak{p}) = \phi^{-1}(\mathfrak{p}')$. This means that $x/1 \in \mathfrak{p}'$, so $\frac{x}{f^n} \in \mathfrak{p}'$ too. So a fraction that belongs to \mathfrak{p} belongs to \mathfrak{p}' . By symmetry the two ideals must be the same.

4. We now know that the map $\psi : \text{Spec } R[f^{-1}] \rightarrow U_f$ is a continuous bijection. It is left to see that it is a homeomorphism. We will show that it is open. In particular, we have to show that a basic open set on the left side is mapped to an open set on the right side. If $y/f^n \in R[f^{-1}]$, we have to show that $U_{y/f^n} \subset \text{Spec } R[f^{-1}]$ has open image under ψ . We'll in fact show what open set it is.

We claim that

$$\psi(U_{y/f^n}) = U_{fy} \subset \text{Spec } R.$$

To see this, \mathfrak{p} is contained in U_{y/f^n} . This means that \mathfrak{p} doesn't contain y/f^n . In particular, \mathfrak{p} doesn't contain the multiple $yf/1$. So $\psi(\mathfrak{p})$ doesn't contain fy . This proves the inclusion \subset .

5. To complete the proof of the claim, and the result, we must show that if $\mathfrak{p} \subset \text{Spec } R[f^{-1}]$ and $\psi(\mathfrak{p}) = \phi^{-1}(\mathfrak{p}) \in U_{fy}$, then y/f^n doesn't belong to \mathfrak{p} . (This is kosher and dandy because we have a bijection.) But the hypothesis implies that $fy \notin \phi^{-1}(\mathfrak{p})$, so $fy/1 \notin \mathfrak{p}$. Dividing by f^{n+1} implies that

$$y/f^n \notin \mathfrak{p}$$

and $\mathfrak{p} \in U_{y/f^n}$. ▲

If $\text{Spec } R$ is a space, and f is thought of as a "function" defined on $\text{Spec } R$, the space U_f is to be thought of as the set of points where f "doesn't vanish" or "is invertible." Thinking about rings in terms of their spectra is a very useful idea. We will bring it up when appropriate.

³Suppose $\frac{x}{f^n} = \frac{y}{f^k}$ for $y \in \mathfrak{p}$. Then there is N such that $f^N(f^k x - f^n y) = 0 \in \mathfrak{p}$; since $y \in \mathfrak{p}$ and $f \notin \mathfrak{p}$, it follows that $x \in \mathfrak{p}$.

Remark The construction $R \rightarrow R[f^{-1}]$ as discussed above is an instance of localization. More generally, we defined $S^{-1}R$ for $S \subset R$ multiplicatively closed. We can thus define maps $\text{Spec } S^{-1}R \rightarrow \text{Spec } R$. To understand $S^{-1}R$, it may help to note that

$$\varinjlim_{f \in S} R[f^{-1}]$$

which is a direct limit of rings where one inverts more and more elements.

As an example, consider $S = R - \mathfrak{p}$ for a prime \mathfrak{p} , and for simplicity that R is countable. We can write $S = S_0 \cup S_1 \cup \dots$, where each S_k is generated by a finite number of elements f_0, \dots, f_k . Then $R_{\mathfrak{p}} = \varinjlim_k S_k^{-1}R$. So we have

$$S^{-1}R = \varinjlim_k R[f_0^{-1}, f_1^{-1}, \dots, f_k^{-1}] = \varinjlim_k R[(f_0 \dots f_k)^{-1}].$$

The functions we invert in this construction are precisely those which do not contain \mathfrak{p} , or where “the functions don’t vanish.”

The geometric idea is that to construct $\text{Spec } S^{-1}R = \text{Spec } R_{\mathfrak{p}}$, we keep cutting out from $\text{Spec } R$ vanishing locuses of various functions that do not intersect \mathfrak{p} . In the end, you don’t restrict to an open set, but to an intersection of them.

EXERCISE 4.8 Say that R is *semi-local* if it has finitely many maximal ideals. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subset R$ be primes. The complement of the union, $S = R \setminus \bigcup \mathfrak{p}_i$, is closed under multiplication, so we can localize. $R[S^{-1}] = R_S$ is called the *semi-localization* of R at the \mathfrak{p}_i .

The result of semi-localization is always semi-local. To see this, recall that the ideals in R_S are in bijection with ideals in R contained in $\bigcup \mathfrak{p}_i$. Now use prime avoidance.

Definition 1.27 For a finitely generated R -module M , define $\mu_R(M)$ to be the smallest number of elements that can generate M .

This is not the same as the cardinality of a minimal set of generators. For example, 2 and 3 are a minimal set of generators for \mathbb{Z} over itself, but $\mu_{\mathbb{Z}}(\mathbb{Z}) = 1$.

Theorem 1.28 Let R be semi-local with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$. Let $k_i = R/\mathfrak{m}_i$. Then

$$\mu_R(M) = \max\{\dim_{k_i} M/\mathfrak{m}_i M\}$$

Proof. **TO BE ADDED:** proof ▲

§2 Nilpotent elements

We will now prove a few general results about nilpotent results in a ring. Topologically, the nilpotents do very little: quotienting by them will not change the Spec. Nonetheless, they carry geometric importance, and one thinks of these nilpotents as “infinitesimal thickenings” (in a sense to be elucidated below).

2.1 The radical of a ring

There is a useful corollary of the analysis in the previous section about the Spec of a ring.

Definition 2.1 $x \in R$ is called **nilpotent** if a power of x is zero. The set of nilpotent elements in R is called the **radical** of R and is denoted $\text{Rad}(R)$ (which is an abuse of notation).

The set of nilpotents is just the radical $\text{Rad}((0))$ of the zero ideal, so it is an ideal. It can vary greatly. A domain clearly has no nonzero nilpotents. On the other hand, many rings do:

Example 2.2 For any $n \geq 2$, the ring $\mathbb{Z}[X]/(X^n)$ has a nilpotent, namely X . The ideal of nilpotent elements is (X) .

It is easy to see that a nilpotent must lie in any prime ideal. The converse is also true by the previous analysis. As a corollary of it, we find in fact:

Corollary 2.3 Let R be a commutative ring. Then the set of nilpotent elements of R is precisely $\bigcap_{\mathfrak{p} \in \text{Spec } R} \mathfrak{p}$.

Proof. Apply Lemma 1.12 to the zero ideal. ▲

We now consider a few examples of nilpotent elements.

Example 2.4 (Nilpotents in polynomial rings) Let us now compute the nilpotent elements in the polynomial $R[x]$. The claim is that a polynomial $\sum_{m=0}^n a_m x^m \in R[x]$ is nilpotent if and only if all the coefficients $a_m \in R$ are nilpotent. That is, $\text{Rad}(R[x]) = (\text{Rad}(R))R[x]$.

If a_0, \dots, a_n are nilpotent, then because the nilpotent elements form an ideal, $f = a_0 + \dots + a_n x^n$ is nilpotent. Conversely, if f is nilpotent, then $f^m = 0$ and thus $(a_n x^n)^m = 0$. Thus $a_n x^n$ is nilpotent, and because the nilpotent elements form an ideal, $f - a_n x^n$ is nilpotent. By induction, $a_i x^i$ is nilpotent for all i , so that all a_i are nilpotent.

Before the next example, we need to define a new notion. We now define a power series ring intuitively in the same way they are used in calculus. In fact, we will use power series rings much the same way we used them in calculus; they will serve as keeping track of fine local data that the Zariski topology might “miss” due to its coarseness.

Definition 2.5 Let R be a ring. The **power series ring** $R[[x]]$ is just the set of all expressions of the form $\sum_{i=0}^{\infty} c_i x^i$. The arithmetic for the power series ring will be done term by term formally (since we have no topology, we can’t consider questions of convergence, though a natural topology can be defined making $R[[x]]$ the *completion* of another ring, as we shall see later).

Example 2.6 (Nilpotence in power series rings) Let R be a ring such that $\text{Rad}(R)$ is a finitely generated ideal. (This is satisfied, e.g., if R is *noetherian*, cf. Chapter 5.) Let us consider the question of how $\text{Rad}(R)$ and $\text{Rad}(R[[x]])$ are related. The claim is that

$$\text{Rad}(R[[x]]) = (\text{Rad}(R))R[[x]].$$

If $f \in R[[x]]$ is nilpotent, say with $f^n = 0$, then certainly $a_0^n = 0$, so that a_0 is nilpotent. Because the nilpotent elements form an ideal, we have that $f - a_0$ is also nilpotent, and hence by induction every coefficient of f must be nilpotent in R . For the converse, let $I = \text{Rad}(R)$. There exists an $N > 0$ such that the ideal power $I^N = 0$ by finite generation. Thus if $f \in IR[[x]]$, then $f^N \in I^N R[[x]] = 0$.

EXERCISE 4.9 Prove that $x \in R$ is nilpotent if and only if the localization R_x is the zero ring.

EXERCISE 4.10 Construct an example where $\text{Rad}(R)R[[x]] \neq \text{Rad}(R[[x]])$. (Hint: consider $R = \mathbb{C}[X_1, X_2, X_3, \dots]/(X_1, X_2^2, X_3^3, \dots)$.)

2.2 Lifting idempotents

If R is a ring, and $I \subset R$ a nilpotent ideal, then we want to think of R/I as somehow close to R . For instance, the inclusion $\text{Spec } R/I \hookrightarrow \text{Spec } R$ is a homeomorphism, and one pictures that $\text{Spec } R$ has some “fuzz” added (with the extra nilpotents in I) that is killed in $\text{Spec } R/I$.

One manifestation of the “closeness” of R and R/I is the following result, which states that the idempotent elements⁴ of the two are in natural bijection. For convenience, we state it in additional generality (that is, for noncommutative rings).

Lemma 2.7 (Lifting idempotents) *Suppose $I \subset R$ is a nilpotent two-sided ideal, for R any⁵ ring. Let $\bar{e} \in R/I$ be an idempotent. Then there is an idempotent $e \in R$ which reduces to \bar{e} .*

Note that if J is a two-sided ideal in a noncommutative ring, then so are the powers of J .

Proof. Let us first assume that $I^2 = 0$. We can find $e_1 \in R$ which reduces to e , but e_1 is not necessarily idempotent. By replacing R with $\mathbb{Z}[e_1]$ and I with $\mathbb{Z}[e_1] \cap I$, we may assume that R is in fact commutative. However,

$$e_1^2 \in e_1 + I.$$

Suppose we want to modify e_1 by i such that $e = e_1 + i$ is idempotent and $i \in I$; then e will do as in the lemma. We would then necessarily have

$$e_1 + i = (e_1 + i)^2 = e_1^2 + 2e_1i \quad \text{as } I^2 = 0.$$

In particular, we must satisfy

$$i(1 - 2e_1) = e_1^2 - e_1 \in I.$$

We claim that $1 - 2e_1 \in R$ is invertible, so that we can solve for $i \in I$. However, R is commutative. It thus suffices to check that $1 - 2e_1$ lies in no maximal ideal of R . But the image of e_1 in R/\mathfrak{m} for any maximal ideal $\mathfrak{m} \subset R$ is either zero or one. So $1 - 2e_1$ has image either 1 or -1 in R/\mathfrak{m} . Thus it is invertible.

This establishes the result when I has zero square. In general, suppose $I^n = 0$. We have the sequence of noncommutative rings:

$$R \twoheadrightarrow R/I^{n-1} \twoheadrightarrow R/I^{n-2} \dots \twoheadrightarrow R/I.$$

The kernel at each step is an ideal whose square is zero. Thus, we can use the lifting idempotents partial result proved above each step of the way and lift $\bar{e} \in R/I$ to some $e \in R$. \blacktriangle

While the above proof has the virtue of applying to noncommutative rings, there is a more conceptual argument for commutative rings. The idea is that idempotents in A measure disconnections of $\text{Spec } A$.⁶ Since the topological space underlying $\text{Spec } A$ is unchanged when one quotients by nilpotents, idempotents are unaffected. We prove:

Proposition 2.8 *If $X = \text{Spec } A$, then there is a one-to-one correspondence between $\text{Idem}(A)$ and the open and closed subsets of X .*

Proof. Suppose I is the radical of (e) for an idempotent $e \in R$. We show that $V(I)$ is open and closed. Since V is unaffected by passing to the radical, we will assume without loss of generality that

$$I = (e).$$

⁴Recall that an element $e \in R$ is idempotent if $e^2 = e$.

⁵Not necessarily commutative.

⁶More generally, in any *ringed space* (a space with a sheaf of rings), the idempotents in the ring of global sections correspond to the disconnections of the topological space.

I claim that $\text{Spec } R - V(I)$ is just $V(1 - e) = V((1 - e))$. This is a closed set, so proving this claim will imply that $V(I)$ is open. Indeed, $V(e) = V((e))$ cannot intersect $V(1 - e)$ because if

$$\mathfrak{p} \in V(e) \cap V(1 - e),$$

then $e, 1 - e \in \mathfrak{p}$, so $1 \in \mathfrak{p}$. This is a contradiction since \mathfrak{p} is necessarily prime.

Conversely, suppose that $\mathfrak{p} \in \text{Spec } R$ belongs to neither $V(e)$ nor $V(1 - e)$. Then $e \notin \mathfrak{p}$ and $1 - e \notin \mathfrak{p}$. So the product

$$e(1 - e) = e - e^2 = 0$$

cannot lie in \mathfrak{p} . But necessarily $0 \in \mathfrak{p}$, contradiction. So $V(e) \cup V(1 - e) = \text{Spec } R$. This implies the claim.

Next, we show that if $V(I)$ is open, then I is the radical of (e) for an idempotent e . For this it is sufficient to prove:

Lemma 2.9 *Let $I \subset R$ be such that $V(I) \subset \text{Spec } R$ is open. Then I is principal, generated by (e) for some idempotent $e \in R$.*

Proof. Suppose that $\text{Spec } R - V(I) = V(J)$ for some ideal $J \subset R$. Then the intersection $V(I) \cap V(J) = V(I + J)$ is all of R , so $I + J$ cannot be a proper ideal (or it would be contained in a prime ideal). In particular, $I + J = R$. So we can write

$$1 = x + y, \quad x \in I, y \in J.$$

Now $V(I) \cup V(J) = V(IJ) = \text{Spec } R$. This implies that every element of IJ is nilpotent by the next lemma. ▲

Lemma 2.10 *Suppose $V(X) = \text{Spec } R$ for $X \subset R$ an ideal. Then every element of X is nilpotent.*

Proof. Indeed, suppose $x \in X$ were non-nilpotent. Then the ring R_x is not the zero ring, so it has a prime ideal. The map $\text{Spec } R_x \rightarrow \text{Spec } R$ is, as discussed in class, a homeomorphism of $\text{Spec } R_x$ onto $D(x)$. So $D(x) \subset \text{Spec } R$ (the collection of primes not containing x) is nonempty. In particular, there is $\mathfrak{p} \in \text{Spec } R$ with $x \notin \mathfrak{p}$, so $\mathfrak{p} \notin V(X)$. So $V(X) \neq \text{Spec } R$, contradiction. ▲

Return to the proof of the main result. We have shown that IJ is nilpotent. In particular, in the expression $x + y = 1$ we had earlier, we have that xy is nilpotent. Say $(xy)^k = 0$. Then expand

$$1 = (x + y)^{2k} = \sum_{i=0}^{2k} \binom{2k}{i} x^i y^{2k-i} = \sum' + \sum''$$

where \sum' is the sum from $i = 0$ to $i = k$ and \sum'' is the sum from $k + 1$ to $2k$. Then $\sum' \sum'' = 0$ because in every term occurring in the expansion, a multiple of $x^k y^k$ occurs. Also, $\sum' \in I$ and $\sum'' \in J$ because $x \in I, y \in J$.

All in all, we find that it is possible to write

$$1 = x' + y', \quad x' \in I, y' \in J, x'y' = 0.$$

(We take $x' = \sum', y' = \sum''$.) Then $x'(1 - x') = 0$ so $x' \in I$ is idempotent. Similarly $y' = 1 - x'$ is. We have that

$$V(I) \subset V(x'), \quad V(J) \subset V(y')$$

and $V(x'), V(y')$ are complementary by the earlier arguments, so necessarily

$$V(I) = V(x'), \quad V(J) = V(y').$$

Since an ideal generated by an idempotent is automatically radical, it follows that:

$$I = (x'), \quad J = (y').$$
▲

There are some useful applications of this in representation theory, because one can look for idempotents in endomorphism rings; these indicate whether a module can be decomposed as a direct sum into smaller parts. Except, of course, that endomorphism rings aren't necessarily commutative and this proof breaks down.

Thus we get:

Proposition 2.11 *Let A be a ring and I a nilpotent ideal. Then $\text{Idem}(A) \rightarrow \text{Idem}(A/I)$ is bijective.*

Proof. Indeed, the topological spaces of $\text{Spec } A$ and $\text{Spec } A/I$ are the same. The result then follows from ??.

2.3 Units

Finally, we make a few remarks on *units* modulo nilideals. It is a useful and frequently used trick that adding a nilpotent does not affect the collection of units. This trick is essentially an algebraic version of the familiar “geometric series;” convergence questions do not appear thanks to nilpotence.

Example 2.12 Suppose u is a unit in a ring R and $v \in R$ is nilpotent; we show that $u + v$ is a unit.

Suppose $ua = 1$ and $v^m = 0$ for some $m > 1$. Then $(u + v) \cdot a(1 - av + (av)^2 - \dots \pm (av)^{m-1}) = (1 - (-av))(1 + (-av) + (-av)^2 + \dots + (-av)^{m-1}) = 1 - (-av)^m = 1 - 0 = 1$, so $u + v$ is a unit.

So let R be a ring, $I \subset R$ a nilpotent ideal of *square zero*. Let R^* denote the group of units in R , as usual, and let $(R/I)^*$ denote the group of units in R/I . We have an exact sequence of abelian groups:

$$0 \rightarrow I \rightarrow R^* \rightarrow (R/I)^* \rightarrow 0$$

where the second map is reduction and the first map sends $i \rightarrow 1 + i$. The hypothesis that $I^2 = 0$ shows that the first map is a homomorphism. We should check that the last map is surjective. But if any $a \in R$ maps to a unit in R/I , it clearly can lie in no prime ideal of R , so is a unit itself.

§3 Vista: sheaves on $\text{Spec } R$

3.1 Presheaves

Let X be a topological space.

Definition 3.1 A **presheaf of sets** \mathcal{F} on X assigns to every open subset $U \subset X$ a set $\mathcal{F}(U)$, and to every inclusion $U \subset V$ a **restriction map** $\text{res}_U^V : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$. The restriction map is required to satisfy:

1. $\text{Res}_U^U = \text{id}_{\mathcal{F}(U)}$ for all open sets U .
2. $\text{Res}_U^W = \text{Res}_U^V \circ \text{Res}_V^W$ if $U \subset V \subset W$.

If the sets $\mathcal{F}(U)$ are all groups (resp. rings), and the restriction maps are morphisms of groups (resp. rings), then we say that \mathcal{F} is a sheaf of groups (resp. rings). Often the restriction of an element $a \in U$ to a subset W is denoted $a|_W$.

A **morphism** of presheaves $\mathcal{F} \rightarrow \mathcal{G}$ is a collection of maps $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for each open set U , that commute with the restriction maps in the obvious way. Thus the collection of presheaves on a topological space forms a category.

One should think of the restriction maps as kind of like restricting the domain of a function. The standard example of presheaves is given in this way, in fact.

Example 3.2 Let X be a topological space, and \mathcal{F} the presheaf assigning to each $U \subset X$ the set of continuous functions $U \rightarrow \mathbb{R}$. The restriction maps come from restricting the domain of a function.

Now, in classical algebraic geometry, there are likely to be more continuous functions in the Zariski topology than one really wants. One wants to focus on functions that are given by polynomial equations.

Example 3.3 Let X be the topological space \mathbb{C}^n with the topology where the closed sets are those defined by the zero loci of polynomials (that is, the topology induced on \mathbb{C}^n from the Zariski topology of $\text{Spec } \mathbb{C}[x_1, \dots, x_n]$ via the canonical imbedding $\mathbb{C}^n \hookrightarrow \text{Spec } \mathbb{C}[x_1, \dots, x_n]$). Then there is a presheaf assigning to each open set U the collection of rational functions defined everywhere on U , with the restriction maps being the obvious ones.

Remark The notion of presheaf thus defined relied very little on the topology of X . In fact, we could phrase it in purely categorical terms. Let \mathcal{C} be the category consisting of open subsets $U \subset X$ and inclusions of open subsets $U \subset U'$. This is a rather simple category (the hom-sets are either empty or consist of one element). Then a *presheaf* is just a contravariant functor from \mathcal{C} to **Sets** (or **Grp**, etc.). A morphism of presheaves is a natural transformation of functors.

In fact, given any category \mathcal{C} , we can define the *category of presheaves* on it to be the category of functors $\mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Set})$. This category is complete and cocomplete (we can calculate limits and colimits “pointwise”), and the Yoneda embedding realizes \mathcal{C} as a full subcategory of it. So if $X \in \mathcal{C}$, we get a presheaf $Y \mapsto \text{Hom}_{\mathcal{C}}(Y, X)$. In general, however, such representable presheaves are rather special; for instance, what do they look like for the category of open sets in a topological space?

3.2 Sheaves

Definition 3.4 Let \mathcal{F} be a presheaf of sets on a topological space X . We call \mathcal{F} a **sheaf** if \mathcal{F} further satisfies the following two “sheaf conditions.”

1. (Separatedness) If U is an open set of X covered by a family of open subsets $\{U_i\}$ and there are two elements $a, b \in \mathcal{F}(U)$ such that $a|_{U_i} = b|_{U_i}$ for all U_i , then $a = b$.
2. (Gluability) If U is an open set of X covered by U_i and there are elements $a_i \in \mathcal{F}(U_i)$ such that $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ for all i and j , then there exists an element $a \in \mathcal{F}(U)$ that restricts to the a_i . Notice that by the first axiom, this element is unique.

A *morphism* of sheaves is just a morphism of presheaves, so the sheaves on a topological space X form a full subcategory of presheaves on X .

The above two conditions can be phrased more compactly as follows. Whenever $\{U_i\}_{i \in I}$ is an open cover of $U \subset X$, we require that the following sequence be an equalizer of sets:

$$\mathcal{F}(U) \rightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i, j \in I} \mathcal{F}(U_i \cap U_j)$$

where the two arrows correspond to the two allowable restriction maps. Similarly, we say that a presheaf of abelian groups (resp. rings) is a **sheaf** if it is a sheaf of sets.

Example 3.5 The example of functions gives an example of a sheaf, because functions are determined by their restrictions to an open cover! Namely, if X is a topological space, and we consider the presheaf

$$U \mapsto \{\text{continuous functions } U \rightarrow \mathbb{R}\},$$

then this is clearly a presheaf, because we can piece together continuous functions in a unique manner.

Example 3.6 Here is a refinement of the above example. Let X be a smooth manifold. For each U , let $\mathcal{F}(U)$ denote the group of smooth functions $U \rightarrow \mathbb{R}$. This is easily checked to be a sheaf.

We could, of course, replace “smooth” by “ C^r ” or by “holomorphic” in the case of a complex manifold.

Remark As remarked above, the notion of presheaf can be defined on any category, and does not really require a topological space. The definition of a sheaf requires a bit more topologically, because the idea that a family $\{U_i\}$ covers an open set U was used inescapably in the definition. The idea of covering required the internal structure of the open sets and was not a purely categorical idea. However, Grothendieck developed a way to axiomatize this, and introduced the idea of a *Grothendieck topology* on a category (which is basically a notion of when a family of maps covers something). On a category with a Grothendieck topology (also known as a *site*), one can define the notion of a sheaf in a similar manner as above. See [Vis08].

There is a process that allows one to take any presheaf and associate a sheaf to it. In some sense, this associated sheaf should also be the best “approximation” of our presheaf with a sheaf. This motivates the following universal property:

Definition 3.7 Let \mathcal{F} be a presheaf. Then \mathcal{F}' is said to be the sheafification of \mathcal{F} if for any sheaf \mathcal{G} and a morphism $\mathcal{F} \rightarrow \mathcal{G}$, there is a unique factorization of this morphism as $\mathcal{F} \rightarrow \mathcal{F}' \rightarrow \mathcal{G}$.

Theorem 3.8 We can construct the sheafification of a presheaf \mathcal{F} as follows: $\mathcal{F}'(U) = \{s : U \rightarrow \coprod_{x \in U} \mathcal{F}_x \mid \text{for all } x \in U, s(x) \in \mathcal{F}_x \text{ and there is a neighborhood } V \subset U \text{ and } t \in \mathcal{F}(V) \text{ such that for all } y \in V, s(y) \text{ is the image of } t \text{ in the local ring } \mathcal{F}_y\}$.

TO BE ADDED: proof

In the theory of schemes, when one wishes to replace polynomial rings over \mathbb{C} (or an algebraically closed field) with arbitrary commutative rings, one must drop the idea that a sheaf is necessarily given by functions. A *scheme* is defined as a space with a certain type of sheaf of rings on it. We shall not define a scheme formally, but show how on the building blocks of schemes—objects of the form $\text{Spec } A$ —a sheaf of rings can be defined.

3.3 Sheaves on $\text{Spec } A$

TO BE ADDED: we need to describe how giving sections over basic open sets gives a presheaf in general.

Proposition 3.9 Let A be a ring and let $X = \text{Spec}(A)$. Then the assignment of the ring A_f to the basic open set X_f defines a presheaf of rings on X .

Proof.

Part (i). If $X_g \subset X_f$ are basic open sets, then there exist $n \geq 1$ and $u \in A$ such that $g^n = uf$.

Proof of part (i). Let $S = \{g^n : n \geq 0\}$ and suppose $S \cap (f) = \emptyset$. Then the extension $(f)^e$ into $S^{-1}A$ is a proper ideal, so there exists a maximal ideal $S^{-1}\mathfrak{p}$ of $S^{-1}A$, where $\mathfrak{p} \cap S = \emptyset$. Since $(f)^e \in S^{-1}\mathfrak{p}$, we see that $f/1 \in S^{-1}\mathfrak{p}$, so $f \in \mathfrak{p}$. But $S \cap \mathfrak{p} = \emptyset$ implies that $g \notin \mathfrak{p}$. This is a contradiction, since then $\mathfrak{p} \in X_g \setminus X_f$.

Part (ii). If $X_g \subset X_f$, then there exists a unique map $\rho : A_f \rightarrow A_g$, called the restriction map, which makes the following diagram commute.

$$\begin{array}{ccc} & A & \\ & \swarrow & \searrow \\ A_f & \longrightarrow & A_g \end{array}$$

Proof of part (ii). Let $n \geq 1$ and $u \in A$ be such that $g^n = uf$ by part (i). Note that in A_g ,

$$(f/1)(u/g^n) = (fu/g^n) = 1/1 = 1$$

which means that f maps to a unit in A_g . Hence every f^m maps to a unit in A_g , so the universal property of A_f yields the desired unique map $\rho : A_f \rightarrow A_g$.

Part (iii). If $X_g = X_f$, then the corresponding restriction $\rho : A_f \rightarrow A_g$ is an isomorphism.

Proof of part (iii). The reverse inclusion yields a $\rho' : A_g \rightarrow A_f$ such that the diagram

$$\begin{array}{ccc} & A & \\ & \swarrow & \searrow \\ A_f & \xrightarrow{\rho} & A_g \\ & \xleftarrow{\rho'} & \end{array}$$

commutes. But since the localization map is epic, this implies that $\rho\rho' = \rho'\rho = \mathbf{1}$.

Part (iv). If $X_h \subset X_g \subset X_f$, then the diagram

$$\begin{array}{ccc} A_f & \longrightarrow & A_h \\ & \searrow & \nearrow \\ & A_g & \end{array}$$

of restriction maps commutes.

Proof of part (iv). Consider the following tetrahedron.

$$\begin{array}{ccc} & A & \\ & \swarrow & \searrow \\ A_f & \cdots \longrightarrow & A_h \\ & \searrow & \nearrow \\ & A_g & \end{array}$$

Except for the base, the commutativity of each face of the tetrahedron follows from the universal property of part (ii). But its easy to see that commutativity of the those faces implies commutativity of the base, which is what we want to show.

Part (v). If $X_{\bar{g}} = X_g \subset X_f = X_{\bar{f}}$, then the diagram

$$\begin{array}{ccc} A_f & \longrightarrow & A_g \\ \downarrow & & \downarrow \\ A_{\bar{f}} & \longrightarrow & A_{\bar{g}} \end{array}$$

of restriction maps commutes. (Note that the vertical maps here are isomorphisms.)

Proof of part (v). By part (iv), the two triangles of

$$\begin{array}{ccc} A_f & \longrightarrow & A_g \\ \downarrow & \searrow & \downarrow \\ A_{\bar{f}} & \longrightarrow & A_{\bar{g}} \end{array}$$

commute. Therefore the square commutes.

Part (vi). Fix a prime ideal \mathfrak{p} in A . Consider the direct system consisting of rings A_f for every $f \notin \mathfrak{p}$ and restriction maps $\rho_{fg} : A_f \rightarrow A_g$ whenever $X_g \subset X_f$. Then $\varinjlim A_f \cong A_{\mathfrak{p}}$.

proof of part (vi). First, note that since $f \notin \mathfrak{p}$ and \mathfrak{p} is prime, we know that $f^m \notin \mathfrak{p}$ for all $m \geq 0$. Therefore the image of f^m under the localization $A \rightarrow A_{\mathfrak{p}}$ is a unit, which means the universal property of A_f yields a unique map $\alpha_f : A_f \rightarrow A_{\mathfrak{p}}$ such that the following diagram commutes.

$$\begin{array}{ccc} & A & \\ & \swarrow & \searrow \\ A_f & \xrightarrow{\alpha_f} & A_{\mathfrak{p}} \end{array}$$

Then consider the following tetrahedron.

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & \downarrow & \searrow & \\ A_f & \cdots & & \cdots & A_h \\ & \swarrow & \downarrow & \searrow & \\ & & A_{\mathfrak{p}} & & \end{array}$$

All faces except the bottom commute by construction, so the bottom face commutes as well. This implies that the α_f commute with the restriction maps, as necessary. Now, to see that $\varinjlim A_f \cong A_{\mathfrak{p}}$, we show that $A_{\mathfrak{p}}$ satisfies the universal property of $\varinjlim A_f$.

Suppose B is a ring and there exist maps $\beta_f : A_f \rightarrow B$ which commute with the restrictions. Define $\beta : A \rightarrow B$ as the composition $A \rightarrow A_f \rightarrow B$. The fact that β is independent of choice of f follows from the commutativity of the following diagram.

$$\begin{array}{ccc} & A & \\ & \swarrow & \searrow \\ A_f & \xrightarrow{\rho_{fg}} & A_g \\ & \swarrow \beta_f & \searrow \beta_g \\ & B & \end{array}$$

Now, for every $f \notin \mathfrak{p}$, we know that $\beta(f)$ must be a unit since $\beta(f) = \beta_f(f/1)$ and $f/1$ is a unit in A_f . Therefore the universal property of $A_{\mathfrak{p}}$ yields a unique map $A_{\mathfrak{p}} \rightarrow B$, which clearly commutes with all the arrows necessary to make $\varinjlim A_f \cong A_{\mathfrak{p}}$. \blacktriangle

Proposition 3.10 *Let A be a ring and let $X = \text{Spec}(A)$. The presheaf of rings \mathcal{O}_X defined on X is a sheaf.*

Proof. The proof proceeds in two parts. Let $(U_i)_{i \in I}$ be a covering of X by basic open sets.

Part 1. If $s \in A$ is such that $s_i := \rho_{X, U_i}(s) = 0$ for all $i \in I$, then $s = 0$.

Proof of part 1. Suppose $U_i = X_{f_i}$. Note that s_i is the fraction $s/1$ in the ring A_{f_i} , so $s_i = 0$ implies that there exists some integer m_i such that $s f_i^{m_i} = 0$. Define $g_i = f_i^{m_i}$, and note that we still have an open cover by sets X_{g_i} since $X_{f_i} = X_{g_i}$ (a prime ideal contains an element if and only if it contains every power of that element). Also $s g_i = 0$, so the fraction $s/1$ is still 0 in the ring A_{g_i} . (Essentially, all we're observing here is that we are free to change representation of the basic open sets in our cover to make notation more convenient).

Since X is quasi-compact, choose a finite subcover $X = X_{g_1} \cup \cdots \cup X_{g_n}$. This means that g_1, \dots, g_n must generate the unit ideal, so there exists some linear combination $\sum x_i g_i = 1$ with $x_i \in A$. But then

$$s = s \cdot 1 = s \left(\sum x_i g_i \right) = \sum x_i (s g_i) = 0.$$

Part 2. Let $s_i \in \mathcal{O}_X(U_i)$ be such that for every $i, j \in I$,

$$\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j).$$

(That is, the collection $(s_i)_{i \in I}$ agrees on overlaps). Then there exists a unique $s \in A$ such that $\rho_{X, U_i}(s) = s_i$ for every $i \in I$.

Proof of part 2. Let $U_i = X_{f_i}$, so that $s_i = a_i / (f_i^{m_i})$ for some integers m_i . As in part 1, we can clean up notation by defining $g_i = f_i^{m_i}$, so that $s_i = a_i / g_i$. Choose a finite subcover $X = X_{g_1} \cup \cdots \cup X_{g_n}$. Then the condition that the cover agrees on overlaps means that

$$\frac{a_i g_j}{g_i g_j} = \frac{a_j g_i}{g_i g_j}$$

for all i, j in the finite subcover. This is equivalent to the existence of some k_{ij} such that

$$(a_i g_j - a_j g_i)(g_i g_j)^{k_{ij}} = 0.$$

Let k be the maximum of all the k_{ij} , so that $(a_i g_j - a_j g_i)(g_i g_j)^k = 0$ for all i, j in the finite subcover. Define $b_i = a_i g_i^k$ and $h_i = g_i^{k+1}$. We make the following observations:

$$b_i h_j - b_j h_i = 0, X_{g_i} = X_{h_i}, \text{ and } s_i = a_i / g_i = b_i / h_i$$

The first observation implies that the X_{h_i} cover X , so the h_i generate the unit ideal. Then there exists some linear combination $\sum x_i h_i = 1$. Define $s = \sum x_i b_i$. I claim that this is the global section that restricts to s_i on the open cover.

The first step is to show that it restricts to s_i on our chosen finite subcover. In other words, we want to show that $s/1 = s_i = b_i / h_i$ in A_{h_i} , which is equivalent to the condition that there exist some l_i such that $(s h_i - b_i) h_i^{l_i} = 0$. But in fact, even $l_i = 0$ works:

$$s h_i - b_i = \left(\sum x_j b_j \right) h_i - b_i \left(\sum x_j h_j \right) = \sum x_j (h_i b_j - b_i h_j) = 0.$$

This shows that s restricts to s_i on each set in our finite subcover. Now we need to show that in fact, it restricts to s_i for all of the sets in our cover. Choose any $j \in I$. Then U_1, \dots, U_n, U_j still cover X , so the above process yields an s' such that s' restricts to s_i for all $i \in \{1, \dots, n, j\}$. But then $s - s'$ satisfies the assumptions of part 1 using the cover $\{U_1, \dots, U_n, U_j\}$, so this means $s = s'$. Hence the restriction of s to U_j is also s_j . \blacktriangle

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CRing Project bibliography

- [AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
- [BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.
- [Bou98] Nicolas Bourbaki. *Commutative algebra. Chapters 1–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.
- [Cam88] Oscar Campoli. A principal ideal domain that is not a euclidean domain. *American Mathematical Monthly*, 95(9):868–871, 1988.
- [CF86] J. W. S. Cassels and A. Fröhlich, editors. *Algebraic number theory*, London, 1986. Academic Press Inc. [Harcourt Brace Jovanovich Publishers]. Reprint of the 1967 original.
- [Cla11] Pete L. Clark. Factorization in euclidean domains. 2011. Available at <http://math.uga.edu/~pete/factorization2010.pdf>.
- [dJea10] Aise Johan de Jong et al. *Stacks Project*. Open source project, available at http://www.math.columbia.edu/algebraic_geometry/stacks-git/, 2010.
- [Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.
- [For91] Otto Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Translated from the 1977 German original by Bruce Gilligan, Reprint of the 1981 English translation.
- [GD] Alexander Grothendieck and Jean Dieudonné. *Éléments de géométrie algébrique*. Publications Mathématiques de l’IHÉS.
- [Ger] Anton Geraschenko (mathoverflow.net/users/1). Is there an example of a formally smooth morphism which is not smooth? MathOverflow. <http://mathoverflow.net/questions/200> (version: 2009-10-08).
- [Gil70] Robert Gilmer. An existence theorem for non-Noetherian rings. *The American Mathematical Monthly*, 77(6):621–623, 1970.
- [Gre97] John Greene. Principal ideal domains are almost euclidean. *The American Mathematical Monthly*, 104(2):154–156, 1997.
- [Gro57] Alexander Grothendieck. Sur quelques points d’algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

- [Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. Available at <http://www.math.cornell.edu/~hatcher/AT/AT.pdf>.
- [Hov07] Mark Hovey. *Model Categories*. American Mathematical Society, 2007.
- [KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.
- [Lan94] Serge Lang. *Algebraic number theory*, volume 110 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1994.
- [Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.
- [Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Ern e, Oxford Science Publications.
- [LR08] T. Y. Lam and Manuel L. Reyes. A prime ideal principle in commutative algebra. *J. Algebra*, 319(7):3006–3027, 2008.
- [Mar02] David Marker. *Model theory*, volume 217 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002. An introduction.
- [Mat80] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.
- [McC76] John McCabe. A note on Zariski’s lemma. *The American Mathematical Monthly*, 83(7):560–561, 1976.
- [Mil80] James S. Milne. * tale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.
- [ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [Per04] Herv e Perdry. An elementary proof of Krull’s intersection theorem. *The American Mathematical Monthly*, 111(4):356–357, 2004.
- [Qui] Daniel Quillen. Homology of commutative rings. Mimeographed notes.
- [Ray70] Michel Raynaud. *Anneaux locaux hens茅liens*. Lecture Notes in Mathematics, Vol. 169. Springer-Verlag, Berlin, 1970.
- [RG71] Michel Raynaud and Laurent Gruson. Crit eres de platitude et de projectivit e. Techniques de “platification” d’un module. *Invent. Math.*, 13:1–89, 1971.
- [Ser65] Jean-Pierre Serre. *Alg ebre locale. Multiplicit es*, volume 11 of *Cours au Coll ege de France, 1957–1958, r edig e par Pierre Gabriel. Seconde  dition, 1965. Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1965.
- [Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

- [Ser09] Jean-Pierre Serre. How to use finite fields for problems concerning infinite fields. 2009. arXiv:0903.0517v2.
- [SGA72] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.
- [SGA03] *Revêtements étales et groupe fondamental (SGA 1)*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960–61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)].
- [Tam94] Günter Tamme. *Introduction to étale cohomology*. Universitext. Springer-Verlag, Berlin, 1994. Translated from the German by Manfred Kolster.
- [Vis08] Angelo Vistoli. Notes on Grothendieck topologies, fibered categories, and descent theory. *Published in FGA Explained*, 2008. arXiv:math/0412512v4.
- [Was97] Lawrence C. Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.
- [Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.