Contents

5 Noetherian rings and modules **3**
1 Basics . . . 3
1.1 The noetherian condition . . . 3
1.2 Stability properties . . . 4
1.3 The basis theorem . . . 6
1.4 Noetherian induction . . . 7
2 Associated primes . . . 8
2.1 The support . . . 8
2.2 Associated primes . . . 8
2.3 Localization and Ass(M) . . . 11
2.4 Associated primes determine the support . . . 12
2.5 Primary modules . . . 14
3 Primary decomposition . . . 15
3.1 Irreducible and coprimary modules . . . 15
3.2 Irreducible and primary decompositions . . . 16
3.3 Uniqueness questions . . . 17
4 Artinian rings and modules . . . 19
4.1 Definitions . . . 19
4.2 The main result . . . 20
4.3 Vista: zero-dimensional non-noetherian rings . . . 22

Chapter 5
Noetherian rings and modules

The finiteness condition of a noetherian ring is necessary for much of commutative algebra; many of the results we prove after this will apply only (or mostly) to the noetherian case. In algebraic geometry, the noetherian condition guarantees that the topological space associated to the ring (the Spec) has all its sets quasi-compact; this condition can be phrased as saying that the space itself is noetherian in a certain sense.

We shall start by proving the basic properties of noetherian rings. These are fairly standard and straightforward; they could have been placed after Chapter 1, in fact. More subtle is the structure theory for finitely generated modules over a noetherian ring. While there is nothing as concrete as there is for PIDs (there, one has a very explicit descrition for the isomorphism classes), one can still construct a so-called "primary decomposition." This will be the primary focus after the basic properties of noetherian rings and modules have been established. Finally, we finish with an important subclass of noetherian rings, the *artinian* ones.

§1 Basics

1.1 The noetherian condition

Definition 1.1 Let R be a commutative ring and M an R-module. We say that M is **noetherian** if every submodule of M is finitely generated.

There is a convenient reformulation of the finiteness hypothesis above in terms of the *ascending chain condition.*

Proposition 1.2 *M is a module over R. The following are equivalent:*

1. *M is noetherian.*
2. *Every chain of submodules $M_0 \subset M_1 \subset \cdots \subset M$, eventually stabilizes at some M_N. (Ascending chain condition.)*
3. *Every nonempty collection of submodules of M has a maximal element.*

Proof. Say M is noetherian and we have such a chain

$$M_0 \subset M_1 \subset \dots .$$

Write

$$M' = \bigcup M_i \subset M,$$

which is finitely generated since M is noetherian. Let it be generated by $x_1, \dots, x_n$. Each of these finitely many elements is in the union, so they are all contained in some M_N. This means that

$$M' \subset M_N, \quad \text{so} \quad M_N = M'$$

and the chain stabilizes.

For the converse, assume the ACC. Let $M' \subset M$ be any submodule. Define a chain of submodules $M_0 \subset M_1 \subset \cdots \subset M'$ inductively as follows. First, just take $M_0 = \{0\}$. Take M_{n+1} to be $M_n + Rx$ for some $x \in M' - M_n$, if such an x exists; if not take $M_{n+1} = M_n$. So M_0 is zero, M_1 is generated by some nonzero element of M', M_2 is M_1 together with some element of M' not in M_1, and so on, until (if ever) the chain stabilizes.

However, by construction, we have an ascending chain, so it stabilizes at some finite place by the ascending chain condition. Thus, at some point, it is impossible to choose something in M' that does not belong to M_N. In particular, M' is generated by N elements, since M_N is and $M' = M_N$. This proves the reverse implication. Thus the equivalence of 1 and 2 is clear. The equivalence of 2 and 3 is left to the reader. ▲

Working with noetherian modules over non-noetherian rings can be a little funny, though, so normally this definition is combined with:

Definition 1.3 The ring R is **noetherian** if R is noetherian as an R-module. Equivalently phrased, R is noetherian if all of its ideals are finitely generated.

We start with the basic examples:

Example 1.4

1. Any field is noetherian. There are two ideals: (1) and (0).
2. Any PID is noetherian: any ideal is generated by one element. So $\mathbb{Z}$ is noetherian.

The first basic result we want to prove is that over a noetherian ring, the noetherian modules are precisely the finitely generated ones. This will follow from Proposition 1.5 in the next subsection. So the defining property of noetherian rings is that a submodule of a finitely generated module is finitely generated. (Compare Proposition 1.8.)

Exercise 5.1 The ring $\mathbb{C}[X_1, X_2, \dots]$ of polynomials in infinitely many variables is not noetherian. Note that the ring itself is finitely generated (by the element 1), but there are ideals that are not finitely generated.

Remark Let R be a ring such that every *prime* ideal is finitely generated. Then R is noetherian. See Corollary 1.19, or prove it as an exercise.

1.2 Stability properties

The class of noetherian rings is fairly robust. If one starts with a noetherian ring, most of the elementary operations one can do to it lead to noetherian rings.

Proposition 1.5 *If*

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of modules, then M is noetherian if and only if M', M'' are.

One direction states that noetherianness is preserved under subobjects and quotients. The other direction states that noetherianness is preserved under extensions.

Proof. If M is noetherian, then every submodule of M' is a submodule of M, so is finitely generated. So M' is noetherian too. Now we show that M'' is noetherian. Let $N \subset M''$ and let $\widetilde{N} \subset M$ the inverse image. Then $\widetilde{N}$ is finitely generated, so N—as the homomorphic image of $\widetilde{N}$—is finitely generated So M'' is noetherian.

Suppose M', M'' noetherian. We prove M noetherian. We verify the ascending chain condition. Consider

$$M_1 \subset M_2 \subset \cdots \subset M.$$

Let M_i'' denote the image of M_i in M'' and let M_i' be the intersection of M_i with M'. Here we think of M' as a submodule of M. These are ascending chains of submodules of M', M'', respectively, so they stabilize by noetherianness. So for some N, we have that $n \geq N$ implies

$$M_n' = M_{n+1}', \quad M_n'' = M_{n+1}''.$$

We claim that this implies, for such n,

$$M_n = M_{n+1}.$$

Indeed, say $x \in M_{n+1} \subset M$. Then x maps into something in $M_{n+1}'' = M_n''$. So there is something in M_n, call it y, such that x, y go to the same thing in M''. In particular,

$$x - y \in M_{n+1}$$

goes to zero in M'', so $x - y \in M'$. Thus $x - y \in M_{n+1}' = M_n'$. In particular,

$$x = (x - y) + y \in M_n' + M_n = M_n.$$

So $x \in M_n$, and

$$M_n = M_{n+1}.$$

This proves the result. ▲

The class of noetherian modules is thus "robust." We can get from that the following.

Proposition 1.6 *If $\phi : A \to B$ is a surjection of commutative rings and A is noetherian, then B is noetherian too.*

Proof. Indeed, B is noetherian as an A-module; indeed, it is the quotient of a noetherian A-module (namely, A). However, it is easy to see that the A-submodules of B are just the B-modules in B, so B is noetherian as a B-module too. So B is noetherian. ▲

We know show that noetherianness of a ring is preserved by localization:

Proposition 1.7 *Let R be a commutative ring, $S \subset R$ a multiplicatively closed subset. If R is noetherian, then $S^{-1}R$ is noetherian.*

I.e., the class of noetherian rings is closed under localization.

Proof. Say $\phi : R \to S^{-1}R$ is the canonical map. Let $I \subset S^{-1}R$ be an ideal. Then $\phi^{-1}(I) \subset R$ is an ideal, so finitely generated. It follows that

$$\phi^{-1}(I)(S^{-1}R) \subset S^{-1}R$$

is finitely generated as an ideal in $S^{-1}R$; the generators are the images of the generators of $\phi^{-1}(I)$.

Now we claim that

$$\phi^{-1}(I)(S^{-1}R) = I.$$

The inclusion $\subset$ is trivial. For the latter inclusion, if $x/s \in I$, then $x \in \phi^{-1}(I)$, so

$$x = (1/s)x \in (S^{-1}R)\phi^{-1}(I).$$

This proves the claim and implies that I is finitely generated. ▲

Let R be a noetherian ring. We now characterize the noetherian R-modules.

Proposition 1.8 *An R-module M is noetherian if and only if M is finitely generated.*

Proof. The only if direction is obvious. A module is noetherian if and only if every submodule is finitely generated.

For the if direction, if M is finitely generated, then there is a surjection of R-modules

$$R^n \to M$$ ▲

where R is noetherian. But R^n is noetherian by Proposition 1.5 because it is a direct sum of copies of R. So M is a quotient of a noetherian module and is noetherian.

1.3 The basis theorem

Let us now prove something a little less formal. This is, in fact, the biggest of the "stability" properties of a noetherian ring: we are going to see that finitely generated algebras over noetherian rings are still noetherian.

Theorem 1.9 (Hilbert basis theorem) *If R is a noetherian ring, then the polynomial ring $R[X]$ is noetherian.*

Proof. Let $I \subset R[X]$ be an ideal. We prove that it is finitely generated. For each $m \in \mathbb{Z}_{\geq 0}$, let $I(n)$ be the collection of elements $a \in R$ consisting of the coefficients of x^n of elements of I of degree $\leq n$. This is an ideal, as is easily seen.

In fact, we claim that

$$I(1) \subset I(2) \subset \dots$$

which follows because if $a \in I(1)$, there is an element $aX + \dots$ in I. Thus $X(aX + \dots) = aX^2 + \dots \in I$, so $a \in I(2)$. And so on.

Since R is noetherian, this chain stabilizes at some $I(N)$. Also, because R is noetherian, each $I(n)$ is generated by finitely many elements $a_{n,1}, \dots, a_{n,m_n} \in I(n)$. All of these come from polynomials $P_{n,i} \in I$ such that $P_{n,i} = a_{n,i}X^n + \dots$.

The claim is that the $P_{n,i}$ for $n \leq N$ and $i \leq m_n$ generate I. This is a finite set of polynomials, so if we prove the claim, we will have proved the basis theorem. Let J be the ideal generated by $\{P_{n,i}, n \leq N, i \leq m_n\}$. We know $J \subset I$. We must prove $I \subset J$.

We will show that any element $P(X) \in I$ of degree n belongs to J by induction on n. The degree is the largest nonzero coefficient. In particular, the zero polynomial does not have a degree, but the zero polynomial is obviously in J.

There are two cases. In the first case, $n \geq N$. Then we write

$$P(X) = aX^n + \dots.$$

By definition, $a \in I(n) = I(N)$ since the chain of ideals $I(n)$ stabilized. Thus we can write a in terms of the generators: $a = \sum a_{N,i}\lambda_i$ for some $\lambda_i \in R$. Define the polynomial

$$Q = \sum \lambda_i P_{N,i} x^{n-N} \in J.$$

Then Q has degree n and the leading term is just a. In particular,

$$P - Q$$

is in I and has degree less than n. By the inductive hypothesis, this belongs to J, and since $Q \in J$, it follows that $P \in J$.

Now consider the case of $n < N$. Again, we write $P(X) = aX^n + \dots$. Then $a \in I(n)$. We can write

$$a = \sum a_{n,i}\lambda_i, \quad \lambda_i \in R.$$

But the $a_{n,i} \in I(n)$. The polynomial

$$Q = \sum \lambda_i P_{n,i}$$

belongs to J since $n < N$. In the same way, $P - Q \in I$ has a lower degree. Induction as before implies that $P \in J$. ▲

Example 1.10 Let k be a field. Then $k[x_1, \dots, x_n]$ is noetherian for any n, by the Hilbert basis theorem and induction on n.

Corollary 1.11 *If R is a noetherian ring and R' a finitely generated R-algebra, then R' is noetherian too.*

Proof. Each polynomial ring $R[X_1, \dots, X_n]$ is noetherian by Theorem 1.9 and an easy induction on n. Consequently, any quotient of a polynomial ring (i.e. any finitely generated R-algebra, such as R') is noetherian. ▲

Example 1.12 Any finitely generated commutative ring R is noetherian. Indeed, then there is a surjection

$$\mathbb{Z}[x_1, \dots, x_n] \twoheadrightarrow R$$

where the x_i get mapped onto generators in R. The former is noetherian by the basis theorem, and R is as a quotient noetherian.

Corollary 1.13 *Any ring R can be obtained as a filtered direct limit of noetherian rings.*

Proof. Indeed, R is the filtered direct limit of its finitely generated subrings. ▲

This observation is sometimes useful in commutative algebra and algebraic geometry, in order to reduce questions about arbitrary commutative rings to noetherian rings. Noetherian rings have strong finiteness hypotheses that let you get numerical invariants that may be useful. For instance, we can do things like inducting on the dimension for noetherian local rings.

Example 1.14 Take $R = \mathbb{C}[x_1, \dots, x_n]$. For any algebraic variety V defined by polynomial equations, we know that V is the vanishing locus of some ideal $I \subset R$. Using the Hilbert basis theorem, we have shown that I is finitely generated. This implies that V can be described by *finitely* many polynomial equations.

1.4 Noetherian induction

The finiteness condition on a noetherian ring allows for "induction" arguments to be made; we shall see examples of this in the future.

Proposition 1.15 (Noetherian Induction Principle) *Let R be a noetherian ring, let $\mathcal{P}$ be a property, and let $\mathcal{F}$ be a family of ideals R. Suppose the inductive step: if all ideals in $\mathcal{F}$ strictly larger than $I \in \mathcal{F}$ satisfy $\mathcal{P}$, then I satisfies $\mathcal{P}$. Then all ideals in $\mathcal{F}$ satisfy $\mathcal{P}$.*

Proof. Assume $\mathcal{F}_{\mathrm{crim}} = \{J \in \mathcal{F} | J \text{ does not satisfy } \mathcal{P}\} \neq \varnothing$. Since R is noetherian, $\mathcal{F}_{\mathrm{crim}}$ has a maximal member I. By maximality, all ideals in $\mathcal{F}$ strictly containing I satisfy $\mathcal{P}$, so I also does by the inductive step. ▲

§2 Associated primes

We shall now begin the structure theory for noetherian modules. The first step will be to associate to each module a collection of primes, called the *associated primes*, which lie in a bigger collection of primes (the *support*) where the localizations are nonzero.

2.1 The support

Let R be a noetherian ring. An R-module M is supposed to be thought of as something like a vector bundle, somehow spread out over the topological space $\operatorname{Spec} R$. If $\mathfrak{p} \in \operatorname{Spec} R$, then let $\kappa(\mathfrak{p}) = \text{fr. field } R/\mathfrak{p}$, which is the residue field of $R_{\mathfrak{p}}$. If M is any R-module, we can consider $M \otimes_R \kappa(\mathfrak{p})$ for each $\mathfrak{p}$; it is a vector space over $\kappa(\mathfrak{p})$. If M is finitely generated, then $M \otimes_R \kappa(\mathfrak{p})$ is a finite-dimensional vector space.

Definition 2.1 Let M be a finitely generated R-module. Then $\operatorname{supp} M$, the **support** of M, is defined to be the set of primes $\mathfrak{p} \in \operatorname{Spec} R$ such that $M \otimes_R \kappa(\mathfrak{p}) \neq 0$.

One is supposed to think of a module M as something like a vector bundle over the topological space $\operatorname{Spec} R$. At each $\mathfrak{p} \in \operatorname{Spec} R$, we associate the vector space $M \otimes_R \kappa(\mathfrak{p})$; this is the "fiber." Of course, the intuition of M's being a vector bundle is somewhat limited, since the fibers do not generally have the same dimension. Nonetheless, we can talk about the support, i.e. the set of spaces where the "fiber" is not zero.

Note that $\mathfrak{p} \in \operatorname{supp} M$ if and only if $M_{\mathfrak{p}} \neq 0$. This is because

$$(M \otimes_R R_{\mathfrak{p}})/(\mathfrak{p}R_{\mathfrak{p}}(M \otimes_R R_{\mathfrak{p}})) = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p})$$

and we can use Nakayama's lemma over the local ring $R_{\mathfrak{p}}$. (We are using the fact that M is finitely generated.)

A vector bundle whose support is empty is zero. Thus the following easy proposition is intuitive:

Proposition 2.2 *$M = 0$ if and only if $\operatorname{supp} M = \emptyset$.*

Proof. Indeed, $M = 0$ if and only if $M_{\mathfrak{p}} = 0$ for all primes $\mathfrak{p} \in \operatorname{Spec} R$. This is equivalent to $\operatorname{supp} M = \emptyset$. ▲

Exercise 5.2 Let $0 \to M' \to M \to M'' \to 0$ be exact. Then

$$\operatorname{supp} M = \operatorname{supp} M' \cup \operatorname{supp} M''.$$

We will see soon that that $\operatorname{supp} M$ is closed in $\operatorname{Spec} R$. One imagines that M lives on this closed subset $\operatorname{supp} M$, in some sense.

2.2 Associated primes

Throughout this section, R is a noetherian ring. The *associated primes* of a module M will refer to primes that arise as the annihilators of elements in M. As we shall see, the support of a module is determined by the associated primes. Namely, the associated primes contain the "generic points" (that is, the minimal primes) of the support. In some cases, however, they may contain more.

TO BE ADDED: We are currently using the notation $\operatorname{Ann}(x)$ for the annihilator of $x \in M$. This has not been defined before. Should we add this in a previous chapter?

Definition 2.3 Let M be a finitely generated R-module. The prime ideal $\mathfrak{p}$ is said to be **associated** to M if there exists an element $x \in M$ such that $\mathfrak{p}$ is the annihilator of x. The set of associated primes is $\operatorname{Ass}(M)$.

Note that the annihilator of an element $x \in M$ is not necessarily prime, but it is possible that the annihilator might be prime, in which case it is associated.

Exercise 5.3 Show that $\mathfrak{p} \in \mathrm{Ass}(M)$ if and only if there is an injection $R/\mathfrak{p} \hookrightarrow M$.

Exercise 5.4 Let $\mathfrak{p} \in \operatorname{Spec} R$. Then $\mathrm{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$.

Example 2.4 Take $R = k[x, y, z]$, where k is an integral domain, and let $I = (x^2 - yz, x(z-1))$. Any prime associated to I must contain I, so let's consider $\mathfrak{p} = (x^2 - yz, z-1) = (x^2 - y, z-1)$, which is $I : x$. It is prime because $R/\mathfrak{p} = k[x]$, which is a domain. To see that $(I : x) \subset \mathfrak{p}$, assume $tx \in I \subset \mathfrak{p}$; since $x \notin \mathfrak{p}$, $t \in p$, as desired.

There are two more associated primes, but we will not find them here.

We shall start by proving that $\mathrm{Ass}(M) \neq \emptyset$ for nonzero modules.

Proposition 2.5 *If $M \neq 0$, then M has an associated prime.*

Proof. Consider the collection of ideals in R that arise as the annihilator of a nonzero element in M. Let $I \subset R$ be a maximal element among this collection. The existence of I is guaranteed thanks to the noetherianness of R. Then $I = \mathrm{Ann}(x)$ for some $x \in M$, so $1 \notin I$ because the annihilator of a nonzero element is not the full ring.

I claim that I is prime, and hence $I \in \mathrm{Ass}(M)$. Indeed, suppose $ab \in I$ where $a, b \in R$. This means that

$$(ab)x = 0.$$

▲

Consider the annihilator $\mathrm{Ann}(bx)$ of bx. This contains the annihilator of x, so I; it also contains a.

There are two cases. If $bx = 0$, then $b \in I$ and we are done. Suppose to the contrary $bx \neq 0$. In this case, $\mathrm{Ann}(bx)$ contains $(a) + I$, which contains I. By maximality, it must happen that $\mathrm{Ann}(bx) = I$ and $a \in I$.

In either case, we find that one of a, b belongs to I, so that I is prime.

Example 2.6 (A module with no associated prime) Without the noetherian hypothesis, Proposition 2.5 is *false*. Consider $R = \mathbb{C}[x_1, x_2, \dots]$, the polynomial ring over $\mathbb{C}$ in infinitely many variables, and the ideal $I = (x_1, x_2^2, x_3^3, \dots) \subset R$. The claim is that

$$\mathrm{Ass}(R/I) = \emptyset.$$

To see this, suppose a prime $\mathfrak{p}$ was the annihilator of some $\overline{f} \in R/I$. Then $\overline{f}$ lifts to $f \in R$; it follows that $\mathfrak{p}$ is precisely the set of $g \in R$ such that $fg \in I$. Now f contains only finitely many of the variables x_i, say $x_1, \dots, x_n$. It is then clear that $x_{n+1}^{n+1} f \in I$ (so $x_{n+1}^{n+1} \in \mathfrak{p}$), but $x_{n+1} f \notin I$ (so $x_{n+1} \notin \mathfrak{p}$). It follows that $\mathfrak{p}$ is not a prime, a contradiction.

We shall now show that the associated primes are finite in number.

Proposition 2.7 *If M is finitely generated, then* $\mathrm{Ass}(M)$ *is finite.*

The idea is going to be to use the fact that M is finitely generated to build M out of finitely many pieces, and use that to bound the number of associated primes to each piece. For this, we need:

Lemma 2.8 *Suppose we have an exact sequence of finitely generated R-modules*

$$0 \to M' \to M \to M'' \to 0.$$

Then

$$\mathrm{Ass}(M') \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$$

Proof. The first claim is obvious. If $\mathfrak{p}$ is the annihilator of in $x \in M'$, it is an annihilator of something in M (namely the image of x), because $M' \to M$ is injective. So $\mathrm{Ass}(M') \subset \mathrm{Ass}(M)$.

The harder direction is the other inclusion. Suppose $\mathfrak{p} \in \mathrm{Ass}(M)$. Then there is $x \in M$ such that $\mathfrak{p} = \mathrm{Ann}(x)$. Consider the submodule $Rx \subset M$. If $Rx \cap M' \neq 0$, then we can choose $y \in Rx \cap M' - \{0\}$. I claim that $\mathrm{Ann}(y) = \mathfrak{p}$ and so $\mathfrak{p} \in \mathrm{Ass}(M')$. To see this, $y = ax$ for some $a \in R$. The annihilator of y is the set of elements $b \in R$ such that

$$abx = 0$$

or, equivalently, the set of $b \in R$ such that $ab \in \mathfrak{p} = \mathrm{Ann}(x)$. But $y = ax \neq 0$, so $a \notin \mathfrak{p}$. As a result, the condition $b \in \mathrm{Ann}(y)$ is the same as $b \in \mathfrak{p}$. In other words,

$$\mathrm{Ann}(y) = \mathfrak{p}$$

which proves the claim.

Suppose now that $Rx \cap M' = 0$. Let $\phi : M \twoheadrightarrow M''$ be the surjection. I claim that $\mathfrak{p} = \mathrm{Ann}(\phi(x))$ and consequently that $\mathfrak{p} \in \mathrm{Ass}(M'')$. The proof is as follows. Clearly $\mathfrak{p}$ annihilates $\phi(x)$ as it annihilates x. Suppose $a \in \mathrm{Ann}(\phi(x))$. This means that $\phi(ax) = 0$, so $ax \in \ker \phi = M'$; but $\ker \phi \cap Rx = 0$. So $ax = 0$ and consequently $a \in \mathfrak{p}$. It follows $\mathrm{Ann}(\phi(x)) = \mathfrak{p}$. ▲

The next step in the proof of Proposition 2.7 is that any finitely generated module admits a filtration each of whose quotients are of a particularly nice form. This result is quite useful and will be referred to in the future.

Proposition 2.9 (Dévissage) *For any finitely generated R-module M, there exists a finite filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

such that the successive quotients M_{i+1}/M_i are isomorphic to various $R/\mathfrak{p}_i$ with the $\mathfrak{p}_i \subset R$ prime.

Proof. Let $M' \subset M$ be maximal among submodules for which such a filtration (ending with M') exists. We would like to show that $M' = M$. Now M' is well-defined since 0 has such a filtration and M is noetherian.

There is a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M' \subset M$$

where the successive quotients, *except* possibly the last M/M', are of the form $R/\mathfrak{p}_i$ for $\mathfrak{p}_i \in \mathrm{Spec}\, R$. If $M' = M$, we are done. Otherwise, consider the quotient $M/M' \neq 0$. There is an associated prime of M/M'. So there is a prime $\mathfrak{p}$ which is the annihilator of $x \in M/M'$. This means that there is an injection

$$R/\mathfrak{p} \hookrightarrow M/M'.$$

Now, take M_{l+1} as the inverse image in M of $R/\mathfrak{p} \subset M/M'$. Then, we can consider the finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{l+1},$$

all of whose successive quotients are of the form $R/\mathfrak{p}_i$; this is because $M_{l+1}/M_l = M_{l+1}/M'$ is of this form by construction. We have thus extended this filtration one step further, a contradiction since M' was assumed to be maximal. ▲

Now we are in a position to meet the goal, and prove that $\mathrm{Ass}(M)$ is always a finite set.

Proof (Proof of Proposition 2.7). Suppose M is finitely generated Take our filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M.$$

By induction, we show that $\mathrm{Ass}(M_i)$ is finite for each i. It is obviously true for $i = 0$. Assume now that $\mathrm{Ass}(M_i)$ is finite; we prove the same for $\mathrm{Ass}(M_{i+1})$. We have an exact sequence

$$0 \to M_i \to M_{i+1} \to R/\mathfrak{p}_i \to 0$$

which implies that, by Lemma 2.8,

$$\mathrm{Ass}(M_{i+1}) \subset \mathrm{Ass}(M_i) \cup \mathrm{Ass}(R/\mathfrak{p}_i) = \mathrm{Ass}(M_i) \cup \{\mathfrak{p}_i\},$$

so $\mathrm{Ass}(M_{i+1})$ is also finite. By induction, it is now clear that $\mathrm{Ass}(M_i)$ is finite for every i.

This proves the proposition; it also shows that the number of associated primes is at most the length of the filtration. ▲

Finally, we characterize the zerodivisors on M in terms of the associated primes. The last characterization of the result will be useful in the future. It implies, for instance, that if R is local and $\mathfrak{m}$ the maximal ideal, then if every element of $\mathfrak{m}$ is a zerodivisor on a finitely generated module M, then $\mathfrak{m} \in \mathrm{Ass}(M)$.

Proposition 2.10 *If M is a finitely generated module over a noetherian ring R, then the zerodivisors on M are the union $\bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$.*

More strongly, if $I \subset R$ is any ideal consisting of zerodivisors on M, then I is contained in an associated prime.

Proof. Any associated prime is an annihilator of some element of M, so it consists of zerodivisors. Conversely, if $a \in R$ annihilates $x \in M$, then a belongs to every associated prime of the nonzero module $Ra \subset M$. (There is at least one by Proposition 2.7.)

For the last statement, we use prime avoidance (**??**): if I consists of zerodivisors, then I is contained in the union $\bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$ by the first part of the proof. This is a finite union by **??**, so prime avoidance implies I is contained one of these primes. ▲

Exercise 5.5 For every module M over any (not necessarily noetherian) ring R, the set of M-zerodivisors$\mathcal{Z}(M)$ is a union of prime ideals. In general, there is an easy characterization of sets Z which are a union of primes: it is exactly when $R \setminus Z$ is a *saturated multiplicative set.* This is Kaplansky's Theorem 2.

Definition 2.11 A multiplicative set $S \neq \varnothing$ is a *saturated multiplicative set* if for all $a, b \in R$, $a, b \in S$ if and only if $ab \in S$. ("multiplicative set" just means the "if" part)

To see that $\mathcal{Z}(M)$ is a union of primes, just verify that its complement is a saturated multiplicative set.

2.3 Localization and $\mathrm{Ass}(M)$

It turns out to be extremely convenient that the construction $M \to \mathrm{Ass}(M)$ behaves about as nicely with respect to localization as we could possibly want. This lets us, in fact, reduce arguments to the case of a local ring, which is a significant simplification.

So, as usual, let R be noetherian, and M a finitely generated R-module. Let further $S \subset R$ be a multiplicative subset. Then $S^{-1}M$ is a finitely generated module over the noetherian ring $S^{-1}M$. So it makes sense to consider both $\mathrm{Ass}(M) \subset \mathrm{Spec}\, R$ and $\mathrm{Ass}(S^{-1}M) \subset \mathrm{Spec}\, S^{-1}R$. But we also know that $\mathrm{Spec}\, S^{-1}R \subset \mathrm{Spec}\, R$ is just the set of primes of R that do not intersect S. Thus, we can directly compare $\mathrm{Ass}(M)$ and $\mathrm{Ass}(S^{-1}M)$, and one might conjecture (correctly, as it happens) that $\mathrm{Ass}(S^{-1}M) = \mathrm{Ass}(M) \cap \mathrm{Spec}\, S^{-1}R$.

Proposition 2.12 *Let R noetherian, M finitely generated and $S \subset R$ multiplicatively closed. Then*
$$\mathrm{Ass}(S^{-1}M) = \left\{ S^{-1}\mathfrak{p} : \mathfrak{p} \in \mathrm{Ass}(M), \mathfrak{p} \cap S = \emptyset \right\}.$$

Proof. We first prove the easy direction, namely that $\mathrm{Ass}(S^{-1}M)$ *contains* primes in $\mathrm{Spec}\, S^{-1}R \cap \mathrm{Ass}(M)$.

Suppose $\mathfrak{p} \in \mathrm{Ass}(M)$ and $\mathfrak{p} \cap S = \emptyset$. Then $\mathfrak{p} = \mathrm{Ann}(x)$ for some $x \in M$. Then the annihilator of $x/1 \in S^{-1}M$ is just $S^{-1}\mathfrak{p}$, as one can directly check. Thus $S^{-1}\mathfrak{p} \in \mathrm{Ass}(S^{-1}M)$. So we get the easy inclusion.

Let us now do the harder inclusion. Call the localization map $R \to S^{-1}R$ as ϕ. Let $\mathfrak{q} \in \mathrm{Ass}(S^{-1}M)$. By definition, this means that $\mathfrak{q} = \mathrm{Ann}(x/s)$ for some $x \in M$, $s \in S$. We want to see that $\phi^{-1}(\mathfrak{q}) \in \mathrm{Ass}(M) \subset \mathrm{Spec}\, R$. By definition $\phi^{-1}(\mathfrak{q})$ is the set of elements $a \in R$ such that
$$\frac{ax}{s} = 0 \in S^{-1}M.$$

In other words, by definition of the localization, this is
$$\phi^{-1}(\mathfrak{q}) = \bigcup_{t \in S} \{a \in R : atx = 0 \in M\} = \bigcup \mathrm{Ann}(tx) \subset R.$$

We know, however, that among elements of the form $\mathrm{Ann}(tx)$, there is a *maximal* element $I = \mathrm{Ann}(t_0x)$ for some $t_0 \in S$, since R is noetherian. The claim is that $I = \phi^{-1}(\mathfrak{q})$, so $\phi^{-1}(\mathfrak{q}) \in \mathrm{Ass}(M)$.

Indeed, any other annihilator $I' = \mathrm{Ann}(tx)$ (for $t \in S$) must be contained in $\mathrm{Ann}(t_0tx)$. However, $I \subset \mathrm{Ann}(t_0x)$ and I is maximal, so $I = \mathrm{Ann}(t_0tx)$ and $I' \subset I$. In other words, I contains all the other annihilators $\mathrm{Ann}(tx)$ for $t \in S$. In particular, the big union above, i.e. $\phi^{-1}(\mathfrak{q})$, is just $I = \mathrm{Ann}(t_0x)$. In particular, $\phi^{-1}(\mathfrak{q}) = \mathrm{Ann}(t_0x)$ is in $\mathrm{Ass}(M)$. This means that every associated prime of $S^{-1}M$ comes from an associated prime of M, which completes the proof. ▲

Exercise 5.6 Show that, if M is a finitely generated module over a noetherian ring, that the map
$$M \to \bigoplus_{\mathfrak{p} \in \mathrm{Ass}(M)} M_{\mathfrak{p}}$$
is injective. Is this true if M is not finitely generated?

2.4 Associated primes determine the support

The next claim is that the support and the associated primes are related.

Proposition 2.13 *The support is the closure of the associated primes:*
$$\mathrm{supp}\, M = \bigcup_{\mathfrak{q} \in \mathrm{Ass}(M)} \overline{\{\mathfrak{q}\}}$$

By definition of the Zariski topology, this means that a prime $\mathfrak{p} \in \mathrm{Spec}\, R$ belongs to $\mathrm{supp}\, M$ if and only if it contains an associated prime.

Proof. First, we show that $\mathrm{supp}(M)$ contains the set of primes $\mathfrak{p} \in \mathrm{Spec}\, R$ containing an associated prime; this will imply that $\mathrm{supp}(M) \supset \bigcup_{\mathfrak{q} \in \mathrm{Ass}(M)} \overline{\{\mathfrak{q}\}}$. So let $\mathfrak{q}$ be an associated prime and $\mathfrak{p} \supset \mathfrak{q}$. We need to show that
$$\mathfrak{p} \in \mathrm{supp}\, M, \text{ i.e. } M_{\mathfrak{p}} \neq 0.$$

But, since $\mathfrak{q} \in \mathrm{Ass}(M)$, there is an injective map
$$R/\mathfrak{q} \hookrightarrow M,$$

so localization gives an injective map

$$(R/\mathfrak{q})_{\mathfrak{p}} \hookrightarrow M_{\mathfrak{p}}.$$

Here, however, the first object $(R/\mathfrak{q})_{\mathfrak{p}}$ is nonzero since nothing nonzero in $R/\mathfrak{q}$ can be annihilated by something outside $\mathfrak{p}$. So $M_{\mathfrak{p}} \neq 0$, and $\mathfrak{p} \in \operatorname{supp} M$.

Let us now prove the converse inclusion. Suppose that $\mathfrak{p} \in \operatorname{supp} M$. We have to show that $\mathfrak{p}$ contains an associated prime. By assumption, $M_{\mathfrak{p}} \neq 0$, and $M_{\mathfrak{p}}$ is a finitely generated module over the noetherian ring $R_{\mathfrak{p}}$. So $M_{\mathfrak{p}}$ has an associated prime. It follows by Proposition 2.12 that $\operatorname{Ass}(M) \cap \operatorname{Spec} R_{\mathfrak{p}}$ is nonempty. Since the primes of $R_{\mathfrak{p}}$ correspond to the primes contained in $\mathfrak{p}$, it follows that there is a prime contained in $\mathfrak{p}$ that lies in $\operatorname{Ass}(M)$. This is precisely what we wanted to prove. ▲

Corollary 2.14 *For M finitely generated,* $\operatorname{supp} M$ *is closed. Further, every minimal element of* $\operatorname{supp} M$ *lies in* $\operatorname{Ass}(M)$.

Proof. Indeed, the above result says that

$$\operatorname{supp} M = \bigcup_{\mathfrak{q} \in \operatorname{Ass}(M)} \overline{\{\mathfrak{q}\}}.$$

Since $\operatorname{Ass}(M)$ is finite, it follows that $\operatorname{supp} M$ is closed. The above equality also shows that any minimal element of $\operatorname{supp} M$ must be an associated prime. ▲

Example 2.15 Corollary 2.14 is *false* for modules that are not finitely generated. Consider for instance the abelian group $\bigoplus_p \mathbb{Z}/p$. The support of this as a $\mathbb{Z}$-module is precisely the set of all closed points (i.e., maximal ideals) of $\operatorname{Spec} \mathbb{Z}$, and is consequently is not closed.

Corollary 2.16 *The ring R has finitely many minimal prime ideals.*

Proof. Clearly, $\operatorname{supp} R = \operatorname{Spec} R$. Thus every prime ideal of R contains an associated prime of R by Proposition 2.13. ▲

So $\operatorname{Spec} R$ is the finite union of the irreducible closed pieces $\overline{\mathfrak{q}}$ if R is noetherian. **TO BE ADDED:** I am not sure if "irreducibility" has already been defined. Check this.

We have just seen that $\operatorname{supp} M$ is a closed subset of $\operatorname{Spec} R$ and is a union of finitely many irreducible subsets. More precisely,

$$\operatorname{supp} M = \bigcup_{\mathfrak{q} \in \operatorname{Ass}(M)} \overline{\{\mathfrak{q}\}}$$

though there might be some redundancy in this expression. Some associated prime might be contained in others.

Definition 2.17 A prime $\mathfrak{p} \in \operatorname{Ass}(M)$ is an **isolated** associated prime of M if it is minimal (with respect to the ordering on $\operatorname{Ass}(M)$); it is **embedded** otherwise.

So the embedded primes are not needed to describe the support of M.

TO BE ADDED: Examples of embedded primes

Remark It follows that in a noetherian ring, every minimal prime consists of zerodivisors. Although we shall not use this in the future, the same is true in non-noetherian rings as well. Here is an argument.

Let R be a ring and $\mathfrak{p} \subset R$ a minimal prime. Then $R_{\mathfrak{p}}$ has precisely one prime ideal. We now use:

Lemma 2.18 *If a ring R has precisely one prime ideal $\mathfrak{p}$, then any $x \in \mathfrak{p}$ is nilpotent.*

Proof. Indeed, it suffices to see that $R_x = 0$ (**??** 4.9 in Chapter 4) if $x \in \mathfrak{p}$. But $\operatorname{Spec} R_x$ consists of the primes of R not containing x. However, there are no such primes. Thus $\operatorname{Spec} R_x = \emptyset$, so $R_x = 0$. ▲

It follows that every element in $\mathfrak{p}$ is a zerodivisor in $R_{\mathfrak{p}}$. As a result, if $x \in \mathfrak{p}$, there is $\frac{s}{t} \in R_{\mathfrak{p}}$ such that $xs/t = 0$ but $\frac{s}{t} \neq 0$. In particular, there is $t' \notin \mathfrak{p}$ with

$$xst' = 0, \quad st' \neq 0,$$

so that x is a zerodivisor.

2.5 Primary modules

A primary modules are ones that has only one associated prime. It is equivalent to say that any homothety is either injective or nilpotent. As we will see in the next section, any module has a "primary decomposition:" in fact, it embeds as a submodule of a sum of primary modules.

Definition 2.19 Let $\mathfrak{p} \subset R$ be prime, M a finitely generated R-module. Then M is **$\mathfrak{p}$-primary** if

$$\operatorname{Ass}(M) = \{\mathfrak{p}\}.$$

A module is **primary** if it is $\mathfrak{p}$-primary for some prime $\mathfrak{p}$, i.e., has precisely one associated prime.

Proposition 2.20 *Let M be a finitely generated R-module. Then M is $\mathfrak{p}$-primary if and only if, for every $m \in M - \{0\}$, the annihilator $\operatorname{Ann}(m)$ has radical $\mathfrak{p}$.*

Proof. We first need a small observation.

Lemma 2.21 *If M is $\mathfrak{p}$-primary, then any nonzero submodule $M' \subset M$ is $\mathfrak{p}$-primary.*

Proof. Indeed, we know that $\operatorname{Ass}(M') \subset \operatorname{Ass}(M)$ by Lemma 2.8. Since $M' \neq 0$, we also know that M' has an associated prime (Proposition 2.5). Thus $\operatorname{Ass}(M') = \{\mathfrak{p}\}$, so M' is $\mathfrak{p}$-primary. ▲

Let us now return to the proof of the main result, Proposition 2.20. Assume first that M is $\mathfrak{p}$-primary. Let $x \in M$, $x \neq 0$. Let $I = \operatorname{Ann}(x)$; we are to show that $\operatorname{Rad}(I) = \mathfrak{p}$. By definition, there is an injection

$$R/I \hookrightarrow M$$

sending $1 \to x$. As a result, R/I is $\mathfrak{p}$-primary by the above lemma. We want to know that $\mathfrak{p} = \operatorname{Rad}(I)$. We saw that the support $\operatorname{supp} R/I = \{\mathfrak{q} : \mathfrak{q} \supset I\}$ is the union of the closures of the associated primes. In this case,

$$\operatorname{supp}(R/I) = \{\mathfrak{q} : \mathfrak{q} \supset \mathfrak{p}\}.$$

But we know that $\operatorname{Rad}(I) = \bigcap_{\mathfrak{q} \supset I} \mathfrak{q}$, which by the above is just $\mathfrak{p}$. This proves that $\operatorname{Rad}(I) = \mathfrak{p}$. We have shown that if R/I is primary, then I has radical $\mathfrak{p}$.

The converse is easy. Suppose the condition holds and $\mathfrak{q} \in \operatorname{Ass}(M)$, so $\mathfrak{q} = \operatorname{Ann}(x)$ for $x \neq 0$. But then $\operatorname{Rad}(\mathfrak{q}) = \mathfrak{p}$, so

$$\mathfrak{q} = \mathfrak{p}$$

and $\operatorname{Ass}(M) = \{\mathfrak{p}\}$. ▲

We have another characterization.

Proposition 2.22 *Let $M \neq 0$ be a finitely generated R-module. Then M is primary if and only if for each $a \in R$, then the homothety $M \stackrel{a}{\to} M$ is either injective or nilpotent.*

Proof. Suppose first that M is $\mathfrak{p}$-primary. Then multiplication by anything in $\mathfrak{p}$ is nilpotent because the annihilator of everything nonzero has radical $\mathfrak{p}$ by Proposition 2.20. But if $a \notin \mathfrak{p}$, then $\mathrm{Ann}(x)$ for $x \in M - \{0\}$ has radical $\mathfrak{p}$ and cannot contain a.

Let us now do the other direction. Assume that every element of a acts either injectively or nilpotently on M. Let $I \subset R$ be the collection of elements $a \in R$ such that $a^n M = 0$ for n large. Then I is an ideal, since it is closed under addition by the binomial formula: if $a, b \in I$ and a^n, b^n act by zero, then $(a+b)^{2n}$ acts by zero as well.

I claim that I is actually prime. If $a, b \notin I$, then a, b act by multiplication injectively on M. So $a : M \to M, b : M \to M$ are injective. However, a composition of injections is injective, so ab acts injectively and $ab \notin I$. So I is prime.

We need now to check that if $x \in M$ is nonzero, then $\mathrm{Ann}(x)$ has radical I. Indeed, if $a \in R$ annihilates x, then the homothety $M \stackrel{a}{\to} M$ cannot be injective, so it must be nilpotent (i.e. in I). Conversely, if $a \in I$, then a power of a is nilpotent, so a power of a must kill x. It follows that $\mathrm{Ann}(x) = I$. Now, by Proposition 2.20, we see that M is I-primary. ▲

We now have this notion of a primary module. The idea is that all the torsion is somehow concentrated in some prime.

Example 2.23 If R is a noetherian ring and $\mathfrak{p} \in \operatorname{Spec} R$, then $R/\mathfrak{p}$ is $\mathfrak{p}$-primary. More generally, if $I \subset R$ is an ideal, then R/I is ideal if and only if $\mathrm{Rad}(I)$ is prime. This follows from Proposition 2.22.

Exercise 5.7 If $0 \to M' \to M \to M'' \to 0$ is an exact sequence with M', M, M'' nonzero and finitely generated, then M is $\mathfrak{p}$-primary if and only if M', M'' are.

Exercise 5.8 Let M be a finitely generated R-module. Let $\mathfrak{p} \in \operatorname{Spec} R$. Show that the sum of two $\mathfrak{p}$-primary submodules is $\mathfrak{p}$-primary. Deduce that there is a $\mathfrak{p}$-primary submodule of M which contains every $\mathfrak{p}$-primary submodule.

Exercise 5.9 (Bourbaki) Let M be a finitely generated R-module. Let $T \subset \mathrm{Ass}(M)$ be a subset of the associated primes. Prove that there is a submodule $N \subset M$ such that

$$\mathrm{Ass}(N) = T, \quad \mathrm{Ass}(M/N) = \mathrm{Ass}(M) - T.$$

§3 Primary decomposition

This is the structure theorem for modules over a noetherian ring, in some sense. Throuoghout, we fix a noetherian ring R.

3.1 Irreducible and coprimary modules

Definition 3.1 Let M be a finitely generated R-module. A submodule $N \subset M$ is **$\mathfrak{p}$-coprimary** if M/N is $\mathfrak{p}$-primary.

Similarly, we can say that $N \subset M$ is **coprimary** if it is $\mathfrak{p}$-coprimary for some $\mathfrak{p} \in \operatorname{Spec} R$.

We shall now show we can represent any submodule of M as an intersection of coprimary submodules. In order to do this, we will define a submodule of M to be *irreducible* if it cannot be written as a nontrivial intersection of submodules of M. It will follow by general nonsense that any submodule is an intersection of irreducible submodueles. We will then see that any irreducible submodule is coprimary.

Definition 3.2 The submomdule $N \subsetneq M$ is **irreducible** if whenever $N = N_1 \cap N_2$ for $N_1, N_2 \subset M$ submodules, then either one of N_1, N_2 equals N. In other words, it is not the intersection of larger submodules.

Proposition 3.3 *An irreducible submodule $N \subset M$ is coprimary.*

Proof. Say $a \in R$. We would like to show that the homothety

$$M/N \stackrel{a}{\to} M/N$$

is either injective or nilpotent. Consider the following submodules of M/N:

$$K(n) = \{x \in M/N : a^n x = 0\}.$$

Then clearly $K(0) \subset K(1) \subset \dots$; this chain stabilizes as the quotient module is noetherian. In particular, $K(n) = K(2n)$ for large n.

It follows that if $x \in M/N$ is divisible by a^n (n large) and nonzero, then $a^n x$ is also nonzero. Indeed, say $x = a^n y \neq 0$; then $y \notin K(n)$, so $a^n x = a^{2n} y \neq 0$ or we would have $y \in K(2n) = K(n)$. In M/N, the submodules

$$a^n(M/N) \cap \ker(a^n)$$

are equal to zero for large n. But our assumption was that N is irreducible. So one of these submodules of M/N is zero. That is, either $a^n(M/N) = 0$ or $\ker a^n = 0$. We get either injectivity or nilpotence on M/N. This proves the result. ▲

3.2 Irreducible and primary decompositions

We shall now show that in a finitely generated module over a noetherian ring, we can write 0 as an intersection of coprimary modules. This decomposition, which is called a *primary decomposition*, will be deduced from purely general reasoning.

Definition 3.4 An **irreducible decomposition** of the module M is a representation $N_1 \cap N_2 \cdots \cap N_k = 0$, where the $N_i \subset M$ are irreducible submodules.

Proposition 3.5 *If M is finitely generated, then M has an irreducible decomposition. There exist finitely many irreducible submodules $N_1, \dots, N_k$ with*

$$N_1 \cap \dots \cap N_k = 0.$$

In other words,

$$M \to \bigoplus M/N_i$$

is injective. So a finitely generated module over a noetherian ring can be imbedded in a direct sum of primary modules, since by Proposition 3.3 the M/N_i are primary.

Proof. This is now purely formal.

Among the submodules of M, some may be expressible as intersections of finitely many irreducibles, while some may not be. Our goal is to show that 0 is such an intersection. Let $M' \subset M$ be a maximal submodule of M such that M' *cannot* be written as such an intersection. If no such M' exists, then we are done, because then 0 can be written as an intersection of finitely many irreducible submodules.

Now M' is not irreducible, or it would be the intersection of one irreducible submodule. It follows M' can be written as $M' = M'_1 \cap M'_2$ for two strictly larger submodules of M. But by maximality, M'_1, M'_2 admit decompositions as intersections of irreducibles. So M' admits such a decomposition as well, a contradiction. ▲

Corollary 3.6 *For any finitely generated M, there exist coprimary submodules $N_1, \dots, N_k \subset M$ such that $N_1 \cap \dots \cap N_k = 0$.*

Proof. Indeed, every irreducible submodule is coprimary. ▲

For any M, we have an **irreducible decomposition**

$$0 = \bigcap N_i$$

for the N_i a finite set of irreducible (and thus coprimary) submodules. This decomposition here is highly non-unique and non-canonical. Let's try to pare it down to something which is a lot more canonical.

The first claim is that we can collect together modules which are coprimary for some prime.

Lemma 3.7 *Let $N_1, N_2 \subset M$ be $\mathfrak{p}$-coprimary submodules. Then $N_1 \cap N_2$ is also $\mathfrak{p}$-coprimary.*

Proof. We have to show that $M/N_1 \cap N_2$ is $\mathfrak{p}$-primary. Indeed, we have an injection

$$M/N_1 \cap N_2 \rightarrowtail M/N_1 \oplus M/N_2$$

which implies that $\mathrm{Ass}(M/N_1 \cap N_2) \subset \mathrm{Ass}(M/N_1) \cup \mathrm{Ass}(M/N_2) = \{\mathfrak{p}\}$. So we are done. ▲

In particular, if we do not want irreducibility but only primariness in the decomposition

$$0 = \bigcap N_i,$$

we can assume that each N_i is $\mathfrak{p}_i$ coprimary for some prime $\mathfrak{p}_i$ with the $\mathfrak{p}_i$ *distinct*.

Definition 3.8 Such a decomposition of zero, where the different modules N_i are $\mathfrak{p}_i$-coprimary for different $\mathfrak{p}_i$, is called a **primary decomposition**.

3.3 Uniqueness questions

In general, primary decomposition is *not* unique. Nonetheless, we shall see that a limited amount of uniqueness does hold. For instance, the primes that occur are determined.

Let M be a finitely generated module over a noetherian ring R, and suppose $N_1 \cap \dots \cap N_k = 0$ is a primary decomposition. Let us assume that the decomposition is *minimal*: that is, if we dropped one of the N_i, the intersection would no longer be zero. This implies that

$$N_i \not\supset \bigcap_{j \neq i} N_j$$

or we could omit one of the N_i. Then the decomposition is called a **reduced primary decomposition**.

Again, what this tells us is that $M \rightarrowtail \bigoplus M/N_i$. What we have shown is that M can be imbedded in a sum of pieces, each of which is $\mathfrak{p}$-primary for some prime, and the different primes are distinct.

This is **not** unique. However,

Proposition 3.9 *The primes $\mathfrak{p}_i$ that appear in a reduced primary decomposition of zero are uniquely determined. They are the associated primes of M.*

Proof. All the associated primes of M have to be there, because we have the injection

$$M \rightarrowtail \bigoplus M/N_i$$

so the associated primes of M are among those of M/N_i (i.e. the $\mathfrak{p}_i$).

The hard direction is to see that each $\mathfrak{p}_i$ is an associated prime. I.e. if M/N_i is $\mathfrak{p}_i$-primary, then $\mathfrak{p}_i \in \mathrm{Ass}(M)$; we don't need to use primary modules except for primes in the associated primes. Here we need to use the fact that our decomposition has no redundancy. Without loss of generality, it suffices to show that $\mathfrak{p}_1$, for instance, belongs to $\mathrm{Ass}(M)$. We will use the fact that

$$N_1 \not\supset N_2 \cap \dots.$$

So this tells us that $N_2 \cap N_3 \cap \dots$ is not equal to zero, or we would have a containment. We have a map

$$N_2 \cap \dots \cap N_k \to M/N_1;$$

it is injective, since the kernel is $N_1 \cap N_2 \cap \dots \cap N_k = 0$ as this is a decomposition. However, M/N_1 is $\mathfrak{p}_1$-primary, so $N_2 \cap \dots \cap N_k$ is $\mathfrak{p}_1$-primary. In particular, $\mathfrak{p}_1$ is an associated prime of $N_2 \cap \dots \cap N_k$, hence of M. ▲

The primes are determined. The factors are not. However, in some cases they are.

Proposition 3.10 *Let $\mathfrak{p}_i$ be a minimal associated prime of M, i.e. not containing any smaller associated prime. Then the submodule N_i corresponding to $\mathfrak{p}_i$ in the reduced primary decomposition is uniquely determined: it is the kernel of*

$$M \to M_{\mathfrak{p}_i}.$$

Proof. We have that $\bigcap N_j = \{0\} \subset M$. When we localize at $\mathfrak{p}_i$, we find that

$$(\bigcap N_j)_{\mathfrak{p}_i} = \bigcap (N_j)_{\mathfrak{p}_i} = 0$$

as localization is an exact functor. If $j \neq i$, then M/N_j is $\mathfrak{p}_j$ primary, and has only $\mathfrak{p}_j$ as an associated prime. It follows that $(M/N_j)_{\mathfrak{p}_i}$ has no associated primes, since the only associated prime could be $\mathfrak{p}_j$, and that's not contained in $\mathfrak{p}_j$. In particular, $(N_j)_{\mathfrak{p}_i} = M_{\mathfrak{p}_i}$.

Thus, when we localize the primary decomposition at $\mathfrak{p}_i$, we get a trivial primary decomposition: most of the factors are the full $M_{\mathfrak{p}_i}$. It follows that $(N_i)_{\mathfrak{p}_i} = 0$. When we draw a commutative diagram

$$\begin{array}{ccc} N_i & \longrightarrow & (N_i)_{\mathfrak{p}_i} = 0 \\ \downarrow & & \downarrow \\ M & \longrightarrow & M_{\mathfrak{p}_i}. \end{array}$$

we find that N_i goes to zero in the localization.

Now if $x \in \ker(M \to M_{\mathfrak{p}_i}$, then $sx = 0$ for some $s \notin \mathfrak{p}_i$. When we take the map $M \to M/N_i$, sx maps to zero; but s acts injectively on M/N_i, so x maps to zero in M/N_i, i.e. is zero in N_i. ▲

This has been abstract, so:

Example 3.11 Let $R = \mathbb{Z}$. Let $M = \mathbb{Z} \oplus \mathbb{Z}/p$. Then zero can be written as

$$\mathbb{Z} \cap \mathbb{Z}/p$$

as submodules of M. But $\mathbb{Z}$ is $\mathfrak{p}$-coprimary, while $\mathbb{Z}/p$ is (0)-coprimary.

This is not unique. We could have considered

$$\{(n,n), n \in \mathbb{Z}\} \subset M.$$

However, the zero-coprimary part has to be the p-torsion. This is because (0) is the minimal ideal.

The decomposition is always unique, in general, if we have no inclusions among the prime ideals. For $\mathbb{Z}$-modules, this means that primary decomposition is unique for torsion modules. Any torsion group is a direct sum of the p-power torsion over all primes p.

EXERCISE 5.10 Suppose R is a noetherian ring and $R_{\mathfrak{p}}$ is a domain for each prime ideal $\mathfrak{p} \subset R$. Then R is a finite direct product $\prod R_i$ for each R_i a domain.

To see this, consider the minimal primes $\mathfrak{p}_i \in \operatorname{Spec} R$. There are finitely many of them, and argue that since every localization is a domain, $\operatorname{Spec} R$ is disconnected into the pieces $V(\mathfrak{p}_i)$. It follows that there is a decomposition $R = \prod R_i$ where $\operatorname{Spec} R_i$ has $\mathfrak{p}_i$ as the unique minimal prime. Each R_i satisfies the same condition as R, so we may reduce to the case of R having a unique minimal prime ideal. In this case, however, R is reduced, so its unique minimal prime ideal must be zero.

§4 Artinian rings and modules

The notion of an *artinian ring* appears to be dual to that of a noetherian ring, since the chain condition is simply reversed in the definition. However, the artinian condition is much stronger than the noetherian one. In fact, artinianness actually implies noetherianness, and much more. Artinian modules over non-artinian rings are frequently of interest as well; for instance, if R is a noetherian ring and $\mathfrak{m}$ is a maximal ideal, then for any finitely generated R-module M, the module $M/\mathfrak{m}M$ is artinian.

4.1 Definitions

Definition 4.1 A commutative ring R is **Artinian** every descending chain of ideals $I_0 \supset I_1 \supset I_2 \supset \dots$ stabilizes.

Definition 4.2 The same definition makes sense for modules. We can define an R-module M to be **Artinian** if every descending chain of submodules stabilizes.

In fact, as we shall see when we study dimension theory, we actually often do want to study artinian modules over non-artinian rings, so this definition is useful.

EXERCISE 5.11 A module is artinian if and only if every nonempty collection of submodules has a minimal element.

EXERCISE 5.12 A ring which is a finite-dimensional algebra over a field is artinian.

Proposition 4.3 *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then M is Artinian if and only if M', M'' are.*

This is proved in the same way as for noetherianness.

Corollary 4.4 *Let R be artinian. Then every finitely generated R-module is artinian.*

Proof. Standard. ▲

4.2 The main result

This definition is obviously dual to the notion of noetherianness, but it is much more restrictive. The main result is:

Theorem 4.5 *A commutative ring R is artinian if and only if:*

1. *R is noetherian.*
2. *Every prime ideal of R is maximal.*[1]

So artinian rings are very simple—small in some sense. They all look kind of like fields.

We shall prove this result in a series of small pieces. We begin with a piece of the forward implication in Theorem 4.5:

Lemma 4.6 *Let R be artinian. Every prime $\mathfrak{p} \subset R$ is maximal.*

Proof. Indeed, if $\mathfrak{p} \subset R$ is a prime ideal, $R/\mathfrak{p}$ is artinian, as it is a quotient of an artinian ring. We want to show that $R/\mathfrak{p}$ is a field, which is the same thing as saying that $\mathfrak{p}$ is maximal. (In particular, we are essentially proving that an artinian *domain* is a field.)

Let $x \in R/\mathfrak{p}$ be nonzero. We have a descending chain

$$R/\mathfrak{p} \supset (x) \supset (x^2) \dots$$

which necessarily stabilizes. Then we have $(x^n) = (x^{n+1})$ for some n. In particular, we have $x^n = yx^{n+1}$ for some $y \in R/\mathfrak{p}$. But x is a nonzerodivisor, and we find $1 = xy$. So x is invertible. Thus $R/\mathfrak{p}$ is a field. ▲

Next, we claim there are only a few primes in an artinian ring:

Lemma 4.7 *If R is artinian, there are only finitely many maximal ideals.*

Proof. Assume otherwise. Then we have an infinite sequence

$$\mathfrak{m}_1, \mathfrak{m}_2, \dots$$

of distinct maximal ideals. Then we have the descending chain

$$R \supset \mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \dots.$$

This necessarily stabilizes. So for some n, we have that $\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n \subset \mathfrak{m}_{n+1}$. However, this means that $\mathfrak{m}_{n+1}$ contains one of the $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ since these are prime ideals (a familiar argument). Maximality and distinctness of the $\mathfrak{m}_i$ give a contradiction. ▲

In particular, we see that $\operatorname{Spec} R$ for an artinian ring is just a finite set. In fact, since each point is closed, as each prime is maximal, the set has the *discrete topology.* As a result, $\operatorname{Spec} R$ for an artinian ring is *Hausdorff*. (There are very few other cases.)

This means that R factors as a product of rings. Whenever $\operatorname{Spec} R$ can be written as a disjoint union of components, there is a factoring of R into a product $\prod R_i$. So $R = \prod R_i$ where each R_i has only one maximal ideal. Each R_i, as a homomorphic image of R, is artinian. We find, as a result,

TO BE ADDED: mention that disconnections of $\operatorname{Spec} R$ are the same thing as idempotents.

Proposition 4.8 *Any artinian ring is a finite product of local artinian rings.*

[1] This is much different from the Dedekind ring condition—there, zero is not maximal. An artinian domain is necessarily a field, in fact.

Now, let us continue our analysis. We may as well assume that we are working with *local* artinian rings R in the future. In particular, R has a unique prime $\mathfrak{m}$, which must be the radical of R as the radical is the intersection of all primes.

We shall now see that the unique prime ideal $\mathfrak{m} \subset R$ is nilpotent by:

Lemma 4.9 *If R is artinian (not necessarily local), then* $\mathrm{Rad}(R)$ *is nilpotent.*

It is, of course, always true that any *element* of the radical $\mathrm{Rad}(R)$ is nilpotent, but it is not true for a general ring R that $\mathrm{Rad}(R)$ is nilpotent as an *ideal*.

Proof. Call $J = \mathrm{Rad}(R)$. Consider the decreasing filtration

$$R \supset J \supset J^2 \supset J^3 \supset \dots.$$

We want to show that this stabilizes at zero. A priori, we know that it stabilizes *somewhere*. For some n, we have

$$J^n = J^{n'}, \quad n' \geq n.$$

Call the eventual stabilization of these ideals I. Consider ideals I' such that

$$II' \neq 0.$$

There are now two cases:

1. No such I' exists. Then $I = 0$, and we are done: the powers of J^n stabilize at zero.
2. Otherwise, there is a *minimal* such I' (minimal for satisfying $II' \neq 0$) as R is artinian. Necessarily I' is nonzero, and furthermore there is $x \in I'$ with $xI \neq 0$.

   It follows by minimality that
   $$I' = (x),$$
   so I' is principal. Then $xI \neq 0$; observe that xI is also $(xI)I$ as $I^2 = I$ from the definition of I. Since $(xI)I \neq 0$, it follows again by minimality that
   $$xI = (x).$$
   Hence, there is $y \in I$ such that $xy = x$; but now, by construction $I \subset J = \mathrm{Rad}(R)$, implying that y is nilpotent. It follows that
   $$x = xy = xy^2 = \cdots = 0$$
   as y is nilpotent. However, $x \neq 0$ as $xI \neq 0$. This is a contradiction, which implies that the second case cannot occur.

We have now proved the lemma. ▲

Finally, we may prove:

Lemma 4.10 *A local artinian ring R is noetherian.*

Proof. We have the filtration $R \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \dots.$ This eventually stabilizes at zero by Lemma 4.9. I claim that R is noetherian as an R-module. To prove this, it suffices to show that $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is noetherian as an R-module. But of course, this is annihilated by $\mathfrak{m}$, so it is really a vector space over the field $R/\mathfrak{m}$. But $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a subquotient of an artinian module, so is artinian itself. We have to show that it is noetherian. It suffices to show now that if k is a field, and V a k-vector space, then TFAE:

1. V is artinian.
2. V is noetherian.
3. V is finite-dimensional.

This is evident by linear algebra. ▲

Now, finally, we have shown that an artinian ring is noetherian. We have to discuss the converse. Let us assume now that R is noetherian and has only maximal prime ideals. We show that R is artinian. Let us consider $\operatorname{Spec} R$; there are only finitely many minimal primes by the theory of associated primes: every prime ideal is minimal in this case. Once again, we learn that $\operatorname{Spec} R$ is finite and has the discrete topology. This means that R is a product of factors $\prod R_i$ where each R_i is a local noetherian ring with a unique prime ideal. We might as well now prove:

Lemma 4.11 *Let $(R, \mathfrak{m})$ be a local noetherian ring with one prime ideal. Then R is artinian.*

Proof. We know that $\mathfrak{m} = \operatorname{rad}(R)$. So $\mathfrak{m}$ consists of nilpotent elements, so by finite generatedness it is nilpotent. Then we have a finite filtration

$$R \supset \mathfrak{m} \supset \cdots \supset \mathfrak{m}^k = 0.$$

Each of the quotients are finite-dimensional vector spaces, so artinian; this implies that R itself is artinian. ▲

Remark Note that artinian implies noetherian! This statement is true for rings (even non-commutative rings), but not for modules. Take the same example $M = \varinjlim \mathbb{Z}/p^n\mathbb{Z}$ over $\mathbb{Z}$. However, there is a module-theoretic statement which is related.

Corollary 4.12 *For a finitely generated module M over any commutative ring R, the following are equivalent.*

1. *M is an artinian module.*
2. *M has finite length (i.e. is noetherian and artinian).*
3. *$R/\operatorname{Ann} M$ is an artinian ring.*

Proof. **TO BE ADDED:** proof ▲

Exercise 5.13 If R is an artinian ring, and S is a finite R-algebra (finite as an R-module), then S is artinian.

Exercise 5.14 Let M be an artinian module over a commutative ring R, $f : M \to M$ an *injective* homomorphism. Show that f is surjective, hence an isomorphism.

4.3 Vista: zero-dimensional non-noetherian rings

Definition 4.13 (von Neumann) An element $a \in R$ is called *von Neumann regular* if there is some $x \in R$ such that $a = axa$.

Definition 4.14 (McCoy) A element $a \in R$ is *π-regular* if some power of a is von Neumann regular.

Definition 4.15 A element $a \in R$ is *strongly π-regular* (in the commutative case) if the chain $aR \supset a^2R \supset a^3R \supset \cdots$ stabilizes.

A ring R is von Neumann regular (resp. (strongly) π-regular) if every element of R is.

Theorem 4.16 (5.2) *For a commutative ring R, the following are equivalent.*

1. $\dim R = 0$.
2. *R is rad-nil (i.e. the Jacobson radical $J(R)$ is the nilradical) and $R/\operatorname{Rad} R$ is von Neumann regular.*
3. *R is strongly π-regular.*
4. *R is π-regular.*

And any one of these implies

5. *Any non-zero-divisor is a unit.*

Proof. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$ and $4 \Rightarrow 5$. We will not do $1 \Rightarrow 2 \Rightarrow 3$ here.

($3 \Rightarrow 4$) Given $a \in R$, there is some n such that $a^n R = a^{n+1} R = a^{2n} R$, which implies that $a^n = a^n x a^n$ for some x.

($4 \Rightarrow 1$) Is $\mathfrak{p}$ maximal? Let $a \notin \mathfrak{p}$. Since a is π-regular, we have $a^n = a^{2n}x$, so $a^n(1 - a^n x) = 0$, so $1 - a^n x \in \mathfrak{p}$. It follows that a has an inverse mod $\mathfrak{p}$.

($4 \Rightarrow 5$) Using $1 - a^n x = 0$, we get an inverse for a. ▲

Example 4.17 Any local rad-nil ring is zero dimensional, since 2 holds. In particular, for a ring S and maximal ideal $\mathfrak{m}$, $R = S/\mathfrak{m}^n$ is zero dimensional because it is a rad-nil local ring.

Example 4.18 (Split-Null Extension) For a ring A and A-module M, let $R = A \oplus M$ with the multiplication $(a, m)(a', m') = (aa', am' + a'm)$ (i.e. take the multiplication on M to be zero). In R, M is an ideal of square zero. (A is called a *retract* of R because it sits in R and can be recovered by quotienting by some complement.) If A is a field, then R is a rad-nil local ring, with maximal ideal M.

CRing Project contents

I Fundamentals **1**

0 Categories **3**

1 Foundations **37**

2 Fields and Extensions **71**

3 Three important functors **93**

II Commutative algebra **131**

4 The Spec of a ring **133**

5 Noetherian rings and modules **157**

6 Graded and filtered rings **183**

7 Integrality and valuation rings **201**

8 Unique factorization and the class group **233**

9 Dedekind domains **249**

10 Dimension theory **265**

11 Completions **293**

12 Regularity, differentials, and smoothness **313**

III Topics **337**

13 Various topics **339**

14 Homological Algebra **353**

15 Flatness revisited **369**

16 Homological theory of local rings **395**

17 Étale, unramified, and smooth morphisms **425**

18 Complete local rings **459**

19 Homotopical algebra **461**

20 GNU Free Documentation License **469**

CRing Project bibliography

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[Bou98] Nicolas Bourbaki. *Commutative algebra. Chapters 1–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[Cam88] Oscar Campoli. A principal ideal domain that is not a euclidean domain. *American Mathematical Monthly*, 95(9):868–871, 1988.

[CF86] J. W. S. Cassels and A. Fröhlich, editors. *Algebraic number theory*, London, 1986. Academic Press Inc. [Harcourt Brace Jovanovich Publishers]. Reprint of the 1967 original.

[Cla11] Pete L. Clark. Factorization in euclidean domains. 2011. Available at `http://math.uga.edu/~pete/factorization2010.pdf`.

[dJea10] Aise Johan de Jong et al. *Stacks Project*. Open source project, available at `http://www.math.columbia.edu/algebraic_geometry/stacks-git/`, 2010.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[For91] Otto Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Translated from the 1977 German original by Bruce Gilligan, Reprint of the 1981 English translation.

[GD] Alexander Grothendieck and Jean Dieudonné. *Élements de géometrie algébrique*. Publications Mathématiques de l'IHÉS.

[Ger] Anton Geraschenko (mathoverflow.net/users/1). Is there an example of a formally smooth morphism which is not smooth? MathOverflow. `http://mathoverflow.net/questions/200` (version: 2009-10-08).

[Gil70] Robert Gilmer. An existence theorem for non-Noetherian rings. *The American Mathematical Monthly*, 77(6):621–623, 1970.

[Gre97] John Greene. Principal ideal domains are almost euclidean. *The American Mathematical Monthly*, 104(2):154–156, 1997.

[Gro57] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002. Available at `http://www.math.cornell.edu/~hatcher/AT/AT.pdf`.

[Hov07] Mark Hovey. *Model Categories.* American Mathematical Society, 2007.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 2006.

[Lan94] Serge Lang. *Algebraic number theory*, volume 110 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1994.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, third edition, 2002.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics.* Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[LR08] T. Y. Lam and Manuel L. Reyes. A prime ideal principle in commutative algebra. *J. Algebra*, 319(7):3006–3027, 2008.

[Mar02] David Marker. *Model theory*, volume 217 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2002. An introduction.

[Mat80] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series.* Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.

[McC76] John McCabe. A note on Zariski's lemma. *The American Mathematical Monthly*, 83(7):560–561, 1976.

[Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series.* Princeton University Press, Princeton, N.J., 1980.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[Per04] Hervé Perdry. An elementary proof of Krull's intersection theorem. *The American Mathematical Monthly*, 111(4):356–357, 2004.

[Qui] Daniel Quillen. Homology of commutative rings. Mimeographed notes.

[Ray70] Michel Raynaud. *Anneaux locaux henséliens.* Lecture Notes in Mathematics, Vol. 169. Springer-Verlag, Berlin, 1970.

[RG71] Michel Raynaud and Laurent Gruson. Critères de platitude et de projectivité. Techniques de "platification" d'un module. *Invent. Math.*, 13:1–89, 1971.

[Ser65] Jean-Pierre Serre. *Algèbre locale. Multiplicités*, volume 11 of *Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel. Seconde édition, 1965. Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1965.

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

[Ser09] Jean-Pierre Serre. How to use finite fields for problems concerning infinite fields. 2009. arXiv:0903.0517v2.

[SGA72] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[SGA03] *Revêtements étales et groupe fondamental (SGA 1).* Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)].

[Tam94] Günter Tamme. *Introduction to étale cohomology.* Universitext. Springer-Verlag, Berlin, 1994. Translated from the German by Manfred Kolster.

[Vis08] Angelo Vistoli. Notes on Grothendieck topologies, fibered categories, and descent theory. *Published in* FGA Explained, 2008. arXiv:math/0412512v4.

[Was97] Lawrence C. Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1997.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994.