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# Chapter 16

## Homological theory of local rings

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We will then apply the general theory to commutative algebra proper. The use of homological machinery provides a new and elegant characterization of regular local rings (among noetherian local rings, they are the ones with finite global dimension) and leads to proofs of several difficult results about them. For instance, we will be able to prove the rather important result (which one repeatedly uses in algebraic geometry) that a regular local ring is a UFD. As another example, the aforementioned criterion leads to a direct proof of the otherwise non-obvious that a localization of a regular local ring at a prime ideal is still a regular local ring.

**Note: right now, the material on regular local rings is still missing! It should be added.**

### §1 Depth

In this section, we first introduce the notion of *depth* for local rings via the Ext functor, and then show that depth can be measured as the length of a maximal *regular sequence*. After this, we study the theory of regular sequences in general (on not-necessarily-local rings), and show that the depth of a module can be bounded in terms of both its dimension and its associated primes.

#### 1.1 Depth over local rings

Throughout, let  $(R, \mathfrak{m})$  be a noetherian local ring. Let  $k = R/\mathfrak{m}$  be the residue field.

Let  $M \neq 0$  be a finitely generated  $R$ -module. We are going to define an arithmetic invariant of  $M$ , called the *depth*, that will measure in some sense the torsion of  $M$ .

**Definition 1.1** The **depth** of  $M$  is equal to the smallest integer  $i$  such that  $\text{Ext}^i(k, M) \neq 0$ . If there is no such integer, we set  $\text{depth } M = \infty$ .

We shall give another characterization of this shortly that makes no reference to Ext functors, and is purely elementary. We will eventually see that there is always such an  $i$  (at least if  $M \neq 0$ ), so  $\text{depth } M < \infty$ .

**Example 1.2 (Depth zero)** Let us characterize when a module  $M$  has depth zero. Depth zero is equivalent to saying that  $\text{Ext}^0(k, M) = \text{Hom}_R(k, M) \neq 0$ , i.e. that there is a nontrivial morphism

$$k \rightarrow M.$$

As  $k = R/\mathfrak{m}$ , the existence of such a map is equivalent to the existence of a nonzero  $x$  such that  $\text{Ann}(x) = \mathfrak{m}$ , i.e.  $\mathfrak{m} \in \text{Ass}(M)$ . So depth zero is equivalent to having  $\mathfrak{m} \in \text{Ass}(M)$ .

Suppose now that  $\text{depth}(M) \neq 0$ . In particular,  $\mathfrak{m} \notin \text{Ass}(M)$ . Since  $\text{Ass}(M)$  is finite, prime avoidance implies that  $\mathfrak{m} \not\subset \bigcup_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}$ . Thus  $\mathfrak{m}$  contains an element which is a nonzerodivisor on  $M$  (see ??). So we find:

**Proposition 1.3**  *$M$  has depth zero iff every element in  $\mathfrak{m}$  is a zerodivisor on  $M$ .*

Now suppose  $\text{depth} M \neq 0$ . There is  $a \in \mathfrak{m}$  which is a nonzerodivisor on  $M$ , i.e. such that there is an exact sequence

$$0 \rightarrow M \xrightarrow{a} M \rightarrow M/aM \rightarrow 0.$$

For each  $i$ , there is an exact sequence in Ext groups:

$$\text{Ext}^{i-1}(k, M/aM) \rightarrow \text{Ext}^i(k, M) \xrightarrow{a} \text{Ext}^i(k, M) \rightarrow \text{Ext}^i(k, M/aM) \rightarrow \text{Ext}^{i+1}(k, M). \quad (16.1)$$

However, the map  $a : \text{Ext}^i(k, M) \rightarrow \text{Ext}^i(k, M)$  is zero as multiplication by  $a$  kills  $k$ . (If  $a$  kills a module  $N$ , then it kills  $\text{Ext}^*(N, M)$  for all  $M$ .) We see from this that

$$\text{Ext}^i(k, M) \hookrightarrow \text{Ext}^i(k, M/aM)$$

is injective, and

$$\text{Ext}^{i-1}(k, M/aM) \twoheadrightarrow \text{Ext}^i(k, M)$$

is surjective.

**Corollary 1.4** *If  $a \in \mathfrak{m}$  is a nonzerodivisor on  $M$ , then*

$$\text{depth}(M/aM) = \text{depth} M - 1.$$

*Proof.* When  $\text{depth} M = \infty$ , this is easy (left to the reader) from the exact sequence. Suppose  $\text{depth}(M) = n$ . We would like to see that  $\text{depth} M/aM = n - 1$ . That is, we want to see that  $\text{Ext}^{n-1}(k, M/aM) \neq 0$ , but  $\text{Ext}^i(k, M/aM) = 0$  for  $i < n - 1$ . This is direct from the sequence (16.1) above. In fact, surjectivity of  $\text{Ext}^{n-1}(k, M/aM) \rightarrow \text{Ext}^n(k, M)$  shows that  $\text{Ext}^{n-1}(k, M/aM) \neq 0$ . Now let  $i < n - 1$ . Then in (16.1),  $\text{Ext}^i(k, M/aM)$  is sandwiched between two zeros, so it is zero.  $\blacktriangle$

The moral of the above discussion is that one quotients out by a nonzerodivisor, the depth drops by one. In fact, we have described a recursive algorithm for computing  $\text{depth}(M)$ .

1. If  $\mathfrak{m} \in \text{Ass}(M)$ , output zero.
2. If  $\mathfrak{m} \notin \text{Ass}(M)$ , choose an element  $a \in \mathfrak{m}$  which is a nonzerodivisor on  $M$ . Output  $\text{depth}(M/aM) + 1$ .

If one wished to apply this in practice, one would probably start by looking for a nonzerodivisor  $a_1 \in \mathfrak{m}$  on  $M$ , and then looking for one on  $M/a_1M$ , etc. From this we make:

**Definition 1.5** Let  $(R, \mathfrak{m})$  be a local noetherian ring,  $M$  a finite  $R$ -module. A sequence  $a_1, \dots, a_n \in \mathfrak{m}$  is said to be  **$M$ -regular** iff:

1.  $a_1$  is a nonzerodivisor on  $M$
2.  $a_2$  is a nonzerodivisor on  $M/a_1M$
3. ...
4.  $a_i$  is a nonzerodivisor on  $M/(a_1, \dots, a_{i-1})M$  for all  $i$ .

A regular sequence  $a_1, \dots, a_n$  is **maximal** if it can be extended no further, i.e. there is no  $a_{n+1}$  such that  $a_1, \dots, a_{n+1}$  is  $M$ -regular.

We now get the promised “elementary” characterization of depth.

**Corollary 1.6** *depth( $M$ ) is the length of every maximal  $M$ -regular sequence. In particular, all  $M$ -regular sequences have the same length.*

*Proof.* If  $a_1, \dots, a_n$  is  $M$ -regular, then

$$\text{depth } M/(a_1, \dots, a_i)M = \text{depth } M - i$$

for each  $i$ , by an easy induction on  $i$  and the Corollary 1.4. From this the result is clear, because depth zero occurs precisely when  $\mathfrak{m}$  is an associated prime (Proposition 1.3). But it is also clear that a regular sequence  $a_1, \dots, a_n$  is maximal precisely when every element of  $\mathfrak{m}$  acts as a zerodivisor on  $M/(a_1, \dots, a_n)M$ , that is,  $\mathfrak{m} \in \text{Ass}(M/(a_1, \dots, a_n)M)$ .  $\blacktriangle$

**Remark** We could *define* the depth via the length of a maximal  $M$ -regular sequence.

Finally, we can bound the depth in terms of the dimension.

**Corollary 1.7** *Let  $M \neq 0$ . Then the depth of  $M$  is finite. In fact,*

$$\text{depth } M \leq \dim M. \tag{16.2}$$

*Proof.* When  $\text{depth } M = 0$ , the assertion is obvious. Otherwise, there is  $a \in \mathfrak{m}$  which is a nonzerodivisor on  $M$ . We know that

$$\text{depth } M/aM = \text{depth } M - 1$$

and (by ??)

$$\dim M/aM = \dim M - 1.$$

By induction on  $\dim M$ , we have that  $\text{depth } M/aM \leq \dim M/aM$ . From this the induction step is clear, because depth and dim both drop by one after quotienting.  $\blacktriangle$

Generally, the depth is not the dimension.

**Example 1.8** Given any  $M$ , adding  $k$  makes the depth zero:  $M \oplus k$  has  $\mathfrak{m}$  as an associated prime. But the dimension does not jump to zero just by adding a copy of  $k$ . If  $M$  is a direct sum of pieces of differing dimensions, then the bound (16.2) does not exhibit equality. In fact, if  $M, M'$  are finitely generated modules, then we have

$$\text{depth } M \oplus M' = \min(\text{depth } M, \text{depth } M'),$$

which follows at once from the definition of depth in terms of vanishing Ext groups.

**EXERCISE 16.1** Suppose  $R$  is a noetherian local ring whose depth (as a module over itself) is zero. If  $R$  is reduced, then  $R$  is a field.

Finally, we include a result that states that the depth does not depend on the ring so much as the module.

**Proposition 1.9 (Depth and change of rings)** *Let  $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  be a morphism of noetherian local rings. Suppose  $M$  is a finitely generated  $S$ -module, which is also finitely generated as an  $R$ -module. Then  $\text{depth}_R M = \text{depth}_S M$ .*

*Proof.* It is clear that we have the inequality  $\text{depth}_R M \leq \text{depth}_S M$ , by the interpretation of depth via regular sequences. Let  $x_1, \dots, x_n \in R$  be a maximal  $M$ -sequence. We need to show that it is a maximal  $M$ -sequence in  $S$  as well. By quotienting, we may replace  $M$  with  $M/(x_1, \dots, x_n)M$ ; we then have to show that if  $M$  has depth zero as an  $R$ -module, it has depth zero as an  $S$ -module.

But then  $\text{Hom}_R(R/\mathfrak{m}, M) \neq 0$ . This is a  $R$ -submodule of  $M$ , consisting of elements killed by  $\mathfrak{m}$ , and in fact it is a  $S$ -submodule. We are going to show that  $\mathfrak{n}$  annihilates some element of it, which will imply that  $\text{depth}_S M = 0$ .

To see this, note that  $\text{Hom}_R(R/\mathfrak{m}, M)$  is artinian as an  $R$ -module (as it is killed by  $\mathfrak{m}$ ). As a result, it is an artinian  $S$ -module, which means it contains  $\mathfrak{n}$  as an associated prime, proving the claim and the result.  $\blacktriangle$

## 1.2 Regular sequences

In the previous subsection, we defined the notion of *depth* of a finitely generated module over a noetherian local ring using the Ext functors. We then showed that the depth was the length of a maximal regular sequence.

Now, although it will not be necessary for the main results in this chapter, we want to generalize this to the case of a non-local ring. Most of the same arguments go through, though there are some subtle differences. For instance, regular sequences remain regular under permutation in the local case, but not in general. Since there will be some repetition, we shall try to be brief.

We start by generalizing the idea of a regular sequence which is not required to be contained in the maximal ideal of a local ring. Let  $R$  be a noetherian ring, and  $M$  a finitely generated  $R$ -module.

**Definition 1.10** A sequence  $x_1, \dots, x_n \in R$  is  **$M$ -regular** (or is an  **$M$ -sequence**) if for each  $k \leq n$ ,  $x_k$  is a nonzerodivisor on the  $R$ -module  $M/(x_1, \dots, x_{k-1})M$  and also  $(x_1, \dots, x_n)M \neq M$ .

So  $x_1$  is a nonzerodivisor on  $M$ , by the first part. That is, the homothety  $M \xrightarrow{x_1} M$  is injective. The last condition is also going to turn out to be necessary for us. In the previous subsection, it was automatic as  $\mathfrak{m}M \neq M$  (unless  $M = 0$ ) by Nakayama's lemma as  $M$  was assumed finitely generated.

The property of being a regular sequence is inherently an inductive one. Note that  $x_1, \dots, x_n$  is a regular sequence on  $M$  if and only if  $x_1$  is a zerodivisor on  $M$  and  $x_2, \dots, x_n$  is an  $M/x_1M$ -sequence.

**Definition 1.11** If  $M$  is an  $R$ -module and  $I \subset R$  an ideal, then we write  $\text{depth}_I M$  for the length of the length-maximizing  $M$ -sequence contained in  $I$ . When  $R$  is local and  $I \subset R$  the maximal ideal, then we just write  $\text{depth } M$  as before.

While we will in fact have a similar characterization of depth in terms of Ext, in this section we *define* it via regular sequences.

**Example 1.12** The basic example one is supposed to keep in mind is the polynomial ring  $R = R_0[x_1, \dots, x_n]$  and  $M = R$ . Then the sequence  $x_1, \dots, x_n$  is regular in  $R$ .

**Example 1.13** Let  $(R, \mathfrak{m})$  be a regular local ring, and let  $x_1, \dots, x_n$  be a regular system of parameters in  $R$  (i.e. a system of generators for  $\mathfrak{m}$  of minimal size). Then we have seen that the  $\{x_i\}$  form a regular sequence on  $R$ , in any order. This is because each quotient  $R/(x_1, \dots, x_i)$  is itself regular, hence a domain.

As before, we have a simple characterization of depth zero:

**Proposition 1.14** *Let  $R$  be noetherian,  $M$  finitely generated. If  $M$  is an  $R$ -module with  $IM \neq M$ , then  $M$  has depth zero if and only if  $I$  is contained in an element of  $\text{Ass}(M)$ .*

*Proof.* This is analogous to Proposition 1.3. Note that an ideal consists of zerodivisors on  $M$  if and only if it is contained in an associated prime (??). ▲

The above proof used ??, a key fact which will be used repeatedly in the sequel. This is one reason the theory of depth works best for finitely generated modules over noetherian rings.

The first observation to make is that regular sequences are *not* preserved by permutation. This is one nice characteristic that we would like but is not satisfied.

**Example 1.15** Let  $k$  be a field. Consider  $R = k[x, y]/((x-1)y, yz)$ . Then  $x, z$  is a regular sequence on  $R$ . Indeed,  $x$  is a nonzerodivisor and  $R/(x) = k[z]$ . However,  $z, x$  is not a regular sequence because  $z$  is a zerodivisor in  $R$ .

Nonetheless, regular sequences *are* preserved by permutation for local rings under suitable noetherian hypotheses:

**Proposition 1.16** *Let  $R$  be a noetherian local ring and  $M$  a finite  $R$ -module. Then if  $x_1, \dots, x_n$  is a  $M$ -sequence contained in the maximal ideal, so is any permutation  $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ .*

*Proof.* It is clearly enough to check this for a transposition. Namely, if we have an  $M$ -sequence

$$x_1, \dots, x_i, x_{i+1}, \dots, x_n$$

we would like to check that so is

$$x_1, \dots, x_{i+1}, x_i, \dots, x_n.$$

It is here that we use the inductive nature. Namely, all we need to do is check that

$$x_{i+1}, x_i, \dots, x_n$$

is regular on  $M/(x_1, \dots, x_{i-1})M$ , since the first part of the sequence will automatically be regular. Now  $x_{i+2}, \dots, x_n$  will automatically be regular on  $M/(x_1, \dots, x_{i+1})M$ . So all we need to show is that  $x_{i+1}, x_i$  is regular on  $M/(x_1, \dots, x_{i-1})M$ .

The moral of the story is that we have reduced to the following lemma.

**Lemma 1.17** *Let  $R$  be a noetherian local ring. Let  $N$  be a finite  $R$ -module and  $a, b \in R$  an  $N$ -sequence contained in the maximal ideal. Then so is  $b, a$ .*

*Proof.* We can prove this as follows. First,  $a$  will be a nonzerodivisor on  $N/bN$ . Indeed, if not then we can write

$$an = bn'$$

for some  $n, n' \in N$  with  $n \notin bN$ . But  $b$  is a nonzerodivisor on  $N/aN$ , which means that  $bn' \in aN$  implies  $n' \in aN$ . Say  $n' = an''$ . So  $an = ban''$ . As  $a$  is a nonzerodivisor on  $N$ , we see that  $n = bn''$ . Thus  $n \in bN$ , contradiction. This part has not used the fact that  $R$  is local.

Now we claim that  $b$  is a nonzerodivisor on  $N$ . Suppose  $n \in N$  and  $bn = 0$ . Since  $b$  is a nonzerodivisor on  $N/aN$ , we have that  $n \in aN$ , say  $n = an'$ . Thus

$$b(an') = a(bn') = 0.$$

The fact that  $N \xrightarrow{a} N$  is injective implies that  $bn' = 0$ . So we can do the same and get  $n' = an''$ ,  $n'' = an^{(3)}$ ,  $n^{(3)} = an^{(4)}$ , and so on. It follows that  $n$  is a multiple of  $a, a^2, a^3, \dots$ , and hence in  $\mathfrak{m}^j N$  for each  $j$  where  $\mathfrak{m} \subset R$  is the maximal ideal. The Krull intersection theorem now implies that  $n = 0$ .

Together, these arguments imply that  $b, a$  is an  $N$ -sequence, proving the lemma. ▲

The proof of the result is now complete. ▲

One might wonder what goes wrong, and why permutations do not preserve regular sequences in general; after all, oftentimes we can reduce results to their analogs for local rings. Yet the fact that regularity is preserved by permutations for local rings does not extend to arbitrary rings. The problem is that regular sequences do *not* localize. Well, they almost do, but the final condition that  $(x_1, \dots, x_n)M \neq M$  doesn't get preserved. We can state:

**Proposition 1.18** *Suppose  $x_1, \dots, x_n$  is an  $M$ -sequence. Let  $N$  be a flat  $R$ -module. Then if  $(x_1, \dots, x_n)M \otimes_R N \neq M \otimes_R N$ , then  $x_1, \dots, x_n$  is an  $M \otimes_R N$ -sequence.*

*Proof.* This is actually very easy now. The fact that  $x_i : M/(x_1, \dots, x_{i-1})M \rightarrow M/(x_1, \dots, x_{i-1})M$  is injective is preserved when  $M$  is replaced by  $M \otimes_R N$  because the functor  $- \otimes_R N$  is exact. ▲

In particular, it follows that if we have a good reason for supposing that  $(x_1, \dots, x_n)M \otimes N \neq M \otimes N$ , then we'll already be done. For instance, if  $N$  is the localization of  $R$  at a prime ideal containing the  $x_i$ . Then we see that automatically  $x_1, \dots, x_n$  is an  $M_{\mathfrak{p}} = M \otimes_R R_{\mathfrak{p}}$ -sequence.

Finally, we have an analog of the previous correspondence between depth and the vanishing of Ext. Since the argument is analogous to Corollary 1.6, we omit it.

**Theorem 1.19** *Let  $R$  be a ring. Suppose  $M$  is an  $R$ -module and  $IM \neq M$ . All maximal  $M$ -sequences in  $I$  have the same length. This length is the smallest value of  $r$  such that  $\text{Ext}^r(R/I, M) \neq 0$ .*

EXERCISE 16.2 Suppose  $I$  is an ideal in  $R$ . Let  $M$  be an  $R$ -module such that  $IM \neq M$ . Show that  $\text{depth}_I M \geq 2$  if and only if the natural map

$$M \simeq \text{Hom}(R, M) \rightarrow \text{Hom}(I, M)$$

is an isomorphism.

### 1.3 Powers of regular sequences

Regular sequences don't necessarily behave well with respect to permutation or localization without additional hypotheses. However, in all cases they behave well with respect to taking powers. The upshot of this is that the invariant called *depth* that we will soon introduce is invariant under passing to the radical.

We shall deduce this from the following easy fact.

**Lemma 1.20** *Suppose we have an exact sequence of  $R$ -modules*

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0.$$

*Suppose the sequence  $x_1, \dots, x_n \in R$  is  $M'$ -regular and  $M''$ -regular. Then it is  $M$ -regular.*

The converse is not true, of course.

*Proof.* Morally, this is the snake lemma. For instance, the fact that multiplication by  $x_1$  is injective on  $M', M''$  implies by the snake diagram that  $M \xrightarrow{x_1} M$  is injective. However, we don't a priori know that a simple inductive argument on  $n$  will work to prove this. The reason is that it needs to be seen that quotienting each term by  $(x_1, \dots, x_{n-1})$  will preserve exactness. However, a general fact will tell us that this is indeed the case. See below.

Anyway, this general fact now lets us induct on  $n$ . If we assume that  $x_1, \dots, x_{n-1}$  is  $M$ -regular, we need only prove that  $x_n : M/(x_1, \dots, x_{n-1})M \rightarrow M/(x_1, \dots, x_{n-1})M$  is injective. (It is



not surjective or the sequence would not be  $M''$ -regular.) But we have the exact sequence by the next lemma,

$$0 \rightarrow M'/(x_1 \dots x_{n-1})M' \rightarrow M/(x_1 \dots x_{n-1})M \rightarrow M''/(x_1 \dots x_{n-1})M'' \rightarrow 0$$

and the injectivity of  $x_n$  on the two ends implies it at the middle by the snake lemma.  $\blacktriangle$

So we need to prove:

**Lemma 1.21** *Suppose  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is a short exact sequence. Let  $x_1, \dots, x_m$  be an  $M''$ -sequence. Then the sequence*

$$0 \rightarrow M'/(x_1 \dots x_m)M' \rightarrow M/(x_1 \dots x_m)M \rightarrow M''/(x_1 \dots x_m)M'' \rightarrow 0$$

*is exact as well.*

One argument here uses the fact that the Tor functors vanish when one has a regular sequence like this. We can give a direct argument.

*Proof.* By induction, this needs only be proved when  $m = 1$ , since we have the recursive description of regular sequences: in general,  $x_2 \dots x_m$  will be regular on  $M''/x_1M''$ . In any case, we have exactness except possibly at the left as the tensor product is right-exact. So let  $m' \in M'$ ; suppose  $m'$  maps to a multiple of  $x_1$  in  $M$ . We need to show that  $m'$  is a multiple of  $x_1$  in  $M'$ .

Suppose  $m'$  maps to  $x_1m$ . Then  $x_1m$  maps to zero in  $M''$ , so by regularity  $m$  maps to zero in  $M''$ . Thus  $m$  comes from something,  $\bar{m}'$ , in  $M'$ . In particular  $m' - x_1\bar{m}'$  maps to zero in  $M$ , so it is zero in  $M'$ . Thus indeed  $m'$  is a multiple of  $x_1$  in  $M'$ .  $\blacktriangle$

With this lemma proved, we can state:

**Proposition 1.22** *Let  $M$  be an  $R$ -module and  $x_1, \dots, x_n$  an  $M$ -sequence. Then  $x_1^{a_1}, \dots, x_n^{a_n}$  is an  $M$ -sequence for any  $a_1, \dots, a_n \in \mathbb{Z}_{>0}$ .*

*Proof.* We will use:

**Lemma 1.23** *Suppose  $x_1, \dots, x_i, \dots, x_n$  and  $x_1, \dots, x'_i, \dots, x_n$  are  $M$ -sequences for some  $M$ . Then so is  $x_1, \dots, x_i x'_i, \dots, x_n$ .*

*Proof.* As usual, we can mod out by  $(x_1 \dots x_{i-1})$  and thus assume that  $i = 1$ . We have to show that if  $x_1, \dots, x_n$  and  $x'_1, \dots, x_n$  are  $M$ -sequences, then so is  $x_1 x'_1, \dots, x_n$ .

We have an exact sequence

$$0 \rightarrow x_1M/x_1x'_1M \rightarrow M/x_1x'_1M \rightarrow M/x_1M \rightarrow 0.$$

Now  $x_2, \dots, x_n$  is regular on the last term by assumption, and also on the first term, which is isomorphic to  $M/x'_1M$  as  $x_1$  acts as a nonzerodivisor on  $M$ . So  $x_2, \dots, x_n$  is regular on both ends, and thus in the middle. This means that

$$x_1x'_1, \dots, x_n$$

is  $M$ -regular. That proves the lemma.  $\blacktriangle$

So we now can prove the proposition. It is trivial if  $\sum a_i = n$  (i.e. if all are 1) it is clear. In general, we can use complete induction on  $\sum a_i$ . Suppose we know the result for smaller values of  $\sum a_i$ . We can assume that some  $a_j > 1$ . Then the sequence

$$x_1^{a_1}, \dots, x_j^{a_j}, \dots, x_n^{a_n}$$

is obtained from the sequences

$$x_1^{a_1}, \dots, x_j^{a_j-1}, \dots, x_n^{a_n}$$

and

$$x_1^{a_1}, \dots, x_j^1, \dots, x_n^{a_n}$$

by multiplying the middle terms. But the complete induction hypothesis implies that both those two sequences are  $M$ -regular, so we can apply the lemma.  $\blacktriangle$

In general, the product of two regular sequences is not a regular sequence. For instance, consider a regular sequence  $x, y$  in some finitely generated module  $M$  over a noetherian local ring. Then  $y, x$  is regular, but the product sequence  $xy, xy$  is *never* regular.

## 1.4 Depth

We make the following definition slightly differently than in the local case:

**Definition 1.24** Suppose  $I$  is an ideal such that  $IM \neq M$ . Then we define the  $I$ -**depth** of  $M$  to be the maximum length of a maximal  $M$ -sequence contained in  $I$ . When  $R$  is a local ring and  $I$  the maximal ideal, then that number is simply called the **depth** of  $M$ .

The **depth** of a proper ideal  $I \subset R$  is its depth on  $R$ .

The definition is slightly awkward, but it turns out that all maximal  $M$ -sequences in  $I$  have the same length, as we saw in Theorem 1.19. So we can use any of them to compute the depth.

The first thing we can prove using the above machinery is that depth is really a “geometric” invariant, in that it depends only on the radical of  $I$ .

**Proposition 1.25** Let  $R$  be a ring,  $I \subset R$  an ideal, and  $M$  an  $R$ -module with  $IM \neq M$ . Then  $\text{depth}_I M = \text{depth}_{\text{Rad}(I)} M$ .

*Proof.* The inequality  $\text{depth}_I M \leq \text{depth}_{\text{Rad}(I)} M$  is trivial, so we need only show that if  $x_1, \dots, x_n$  is an  $M$ -sequence in  $\text{Rad}(I)$ , then there is an  $M$ -sequence of length  $n$  in  $I$ . For this we just take a high power

$$x_1^N, \dots, x_n^N$$

where  $N$  is large enough such that everything is in  $I$ . We can do this as powers of  $M$ -sequences are  $M$ -sequences (Proposition 1.22).  $\blacktriangle$

This was a fairly easy consequence of the above result on powers of regular sequences. On the other hand, we want to give another proof, because it will let us do more. Namely, we will show that depth is really a function of prime ideals.

For convenience, we set the following condition: if  $IM = M$ , we define

$$\text{depth}_I(M) = \infty.$$

**Proposition 1.26** Let  $R$  be a noetherian ring,  $I \subset R$  an ideal, and  $M$  a finitely generated  $R$ -module. Then

$$\text{depth}_I M = \min_{\mathfrak{p} \in V(I)} \text{depth}_{\mathfrak{p}} M.$$

So the depth of  $I$  on  $M$  can be calculated from the depths at each prime containing  $I$ . In this sense, it is clear that  $\text{depth}_I(M)$  depends only on  $V(I)$  (and the depths on those primes), so clearly it depends only on  $I$  up to radical.

*Proof.* In this proof, we shall use the fact that the length of every maximal  $M$ -sequence is the same (Theorem 1.19).

It is obvious that we have an inequality

$$\text{depth}_I M \leq \min_{\mathfrak{p} \in V(I)} \text{depth}_{\mathfrak{p}} M$$

as each of those primes contains  $I$ . We are to prove that there is a prime  $\mathfrak{p}$  containing  $I$  with

$$\text{depth}_I M = \text{depth}_{\mathfrak{p}} M.$$

But we shall actually prove the stronger statement that there is  $\mathfrak{p} \supset I$  with  $\text{depth}_{\mathfrak{p}} M_{\mathfrak{p}} = \text{depth}_I M$ . Note that localization at a prime can only increase depth because an  $M$ -sequence in  $\mathfrak{p}$  leads to an  $M$ -sequence in  $M_{\mathfrak{p}}$  thanks to Nakayama's lemma and the flatness of localization.

So let  $x_1, \dots, x_n \in I$  be a  $M$ -sequence of maximum length. Then  $I$  acts by zerodivisors on  $M/(x_1, \dots, x_n)M$  or we could extend the sequence further. In particular,  $I$  is contained in an associated prime of  $M/(x_1, \dots, x_n)M$  by elementary commutative algebra (basically, prime avoidance).

Call this associated prime  $\mathfrak{p} \in V(I)$ . Then  $\mathfrak{p}$  is an associated prime of  $M_{\mathfrak{p}}/(x_1, \dots, x_n)M_{\mathfrak{p}}$ , and in particular acts only by zerodivisors on this module. Thus the  $M_{\mathfrak{p}}$ -sequence  $x_1, \dots, x_n$  can be extended no further in  $\mathfrak{p}$ . In particular, since the depth can be computed as the length of *any* maximal  $M_{\mathfrak{p}}$ -sequence,

$$\text{depth}_{\mathfrak{p}} M_{\mathfrak{p}} = \text{depth}_I M. \quad \blacktriangle$$

Perhaps we should note a corollary of the argument above:

**Corollary 1.27** *Hypotheses as above, we have  $\text{depth}_I M \leq \text{depth}_{\mathfrak{p}} M_{\mathfrak{p}}$  for any prime  $\mathfrak{p} \supset I$ . However, there is at least one  $\mathfrak{p} \supset I$  where equality holds.*

We are thus reduced to analyzing depth in the local case.

**EXERCISE 16.3** If  $(R, \mathfrak{m})$  is a local noetherian ring and  $M$  a finitely generated  $R$ -module, then show that  $\text{depth} M = \text{depth}_{\hat{R}} \hat{M}$ , where  $\hat{M}$  is the  $\mathfrak{m}$ -adic completion. (Hint: use  $\hat{M} = M \otimes_R \hat{R}$ , and the fact that  $\hat{R}$  is flat over  $R$ .)

## 1.5 Depth and dimension

Consider an  $R$ -module  $M$ , which is always assumed to be finitely generated. Let  $I \subset R$  be an ideal with  $IM \neq M$ . We deduce from the previous subsections:

**Proposition 1.28** *Let  $M$  be a finitely generated module over the noetherian ring  $R$ . Then*

$$\text{depth}_I M \leq \dim M$$

*for any ideal  $I \subset R$  with  $IM \neq M$ .*

*Proof.* We have proved this when  $R$  is a *local* ring (Corollary 1.7). Now we just use Corollary 1.27 to reduce to the local case.  $\blacktriangle$

This does not tell us much about how  $\text{depth}_I M$  depends on  $I$ , though; it just says something about how it depends on  $M$ . In particular, it is not very helpful when trying to estimate  $\text{depth} I = \text{depth}_I R$ . Nonetheless, there is a somewhat stronger result, which we will need in the future. We start by stating the version in the local case.

**Proposition 1.29** *Let  $(R, \mathfrak{m})$  be a noetherian local ring. Let  $M$  be a finite  $R$ -module. Then the depth of  $\mathfrak{m}$  on  $M$  is at most the dimension of  $R/\mathfrak{p}$  for  $\mathfrak{p}$  an associated prime of  $M$ :*

$$\text{depth } M \leq \min_{\mathfrak{p} \in \text{Ass}(M)} \dim R/\mathfrak{p}.$$

This is sharper than the bound  $\text{depth } M \leq \dim M$ , because each  $\dim R/\mathfrak{p}$  is at most  $\dim M$  (by definition).

*Proof.* To prove this, first assume that the depth is zero. In that case, the result is immediate. We shall now argue inductively. Assume that this is true for modules of smaller depth. We will quotient out appropriately to shrink the support and change the associated primes. Namely, choose a  $M$ -regular (nonzerodivisor on  $M$ )  $x \in R$ . Then  $\text{depth } M/xM = \text{depth } M - 1$ .

Let  $\mathfrak{p}_0$  be an associated prime of  $M$ . We claim that  $\mathfrak{p}_0$  is *properly* contained in an associated prime of  $M/xM$ . We will prove this below. Thus  $\mathfrak{p}_0$  is properly contained in some  $\mathfrak{q}_0 \in \text{Ass}(M/xM)$ .

Now we know that  $\text{depth } M/xM = \text{depth } M - 1$ . Also, by the inductive hypothesis, we know that  $\dim R/\mathfrak{q}_0 \geq \text{depth } M/xM = \text{depth } M - 1$ . But the dimension of  $R/\mathfrak{q}_0$  is strictly smaller than that of  $R/\mathfrak{p}_0$ , so at least  $\dim R/\mathfrak{p}_0 + 1 \geq \text{depth } M$ . This proves the lemma, modulo the result:

**Lemma 1.30** *Let  $(R, \mathfrak{m})$  be a noetherian local ring. Let  $M$  be a finitely generated  $R$ -module,  $x \in \mathfrak{m}$  an  $M$ -regular element. Then each element of  $\text{Ass}(M)$  is properly contained in an element of  $\text{Ass}(M/xM)$ .*

So if we quotient by a regular element, we can make the associated primes jump up.

*Proof.* Let  $\mathfrak{p}_0 \in \text{Ass}(M)$ ; we want to show  $\mathfrak{p}_0$  is properly contained in something in  $\text{Ass}(M/xM)$ .

Indeed,  $x \notin \mathfrak{p}_0$ , so  $\mathfrak{p}_0$  cannot itself be an associated prime. However,  $\mathfrak{p}_0$  annihilates a nonzero element of  $M/xM$ . To see this, consider a maximal principal submodule of  $M$  annihilated by  $\mathfrak{p}_0$ . Let this submodule be  $Rz$  for some  $z \in M$ . Then if  $z$  is a multiple of  $x$ , say  $z = xz'$ , then  $Rz'$  would be a larger submodule of  $M$  annihilated by  $\mathfrak{p}_0$ —here we are using the fact that  $x$  is a nonzerodivisor on  $M$ . So the image of this  $z$  in  $M/xM$  is nonzero and is clearly annihilated by  $\mathfrak{p}_0$ . It follows  $\mathfrak{p}_0$  is contained in an element of  $\text{Ass}(M/xM)$ , necessarily properly.  $\blacktriangle$

EXERCISE 16.4 Another argument for Lemma 1.30 is given in §16 of [GD], vol. IV, by reducing to the coprimary case. Here is a sketch.

The strategy is to use the existence of an exact sequence

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

with  $\text{Ass}(M'') = \text{Ass}(M) - \{\mathfrak{p}_0\}$  and  $\text{Ass}(M') = \{\mathfrak{p}_0\}$ . Quotienting by  $x$  preserves exactness, and we get

$$0 \rightarrow M'/xM' \rightarrow M/xM \rightarrow M''/xM'' \rightarrow 0.$$

Now  $\mathfrak{p}_0$  is properly contained in every associated prime of  $M'/xM'$  (as it acts nilpotently on  $M'$ ). It follows that any element of  $\text{Ass}(M'/xM') \subset \text{Ass}(M/xM)$  will do the job.

In essence, the point is that the result is *trivial* when  $\text{Ass}(M) = \{\mathfrak{p}_0\}$ .

EXERCISE 16.5 Here is a simpler argument for Lemma 1.30, following [Ser65]. Let  $\mathfrak{p}_0 \in \text{Ass}(M)$ , as before. Again as before, we want to show that  $\text{Hom}_R(R/\mathfrak{p}_0, M/xM) \neq 0$ . But we have an exact sequence

$$0 \rightarrow \text{Hom}_R(R/\mathfrak{p}_0, M) \xrightarrow{x} \text{Hom}_R(R/\mathfrak{p}_0, M) \rightarrow \text{Hom}_R(R/\mathfrak{p}_0, M/xM),$$

and since the first map is not surjective (by Nakayama), the last object is nonzero.

Finally, we can globalize the results:

**Proposition 1.31** *Let  $R$  be a noetherian ring,  $I \subset R$  an ideal, and  $M$  a finitely generated module. Then  $\text{depth}_I M$  is at most the length of every chain of primes in  $\text{Spec} R$  that starts at an associated prime of  $M$  and ends at a prime containing  $I$ .*

*Proof.* Currently omitted. ▲

## §2 Cohen-Macaulayness

### 2.1 Cohen-Macaulay modules over a local ring

For a local noetherian ring, we have discussed two invariants of a module: dimension and depth. They generally do not coincide, and Cohen-Macaulay modules will be those where they do.

Let  $(R, \mathfrak{m})$  be a noetherian local ring.

**Definition 2.1** A finitely generated  $R$ -module  $M$  is **Cohen-Macaulay** if  $\text{depth } M = \dim M$ . The ring  $R$  is called **Cohen-Macaulay** if it is Cohen-Macaulay as a module over itself.

We already know that the inequality  $\leq$  always holds. If there is a system of parameters for  $M$  (i.e., a sequence  $x_1, \dots, x_r \in \mathfrak{m}$  such that  $M/(x_1, \dots, x_r)M$  is artinian) which is a regular sequence on  $M$ , then  $M$  is Cohen-Macaulay: we see in fact that  $\dim M = \text{depth } M = r$ . This is the distinguishing trait of Cohen-Macaulay rings.

Let us now give a few examples:

**Example 2.2 (Regular local rings are Cohen-Macaulay)** If  $R$  is regular, then  $\text{depth } R = \dim R$ , so  $R$  is Cohen-Macaulay.

Indeed, we have seen that if  $x_1, \dots, x_n$  is a regular system of parameters for  $R$  (i.e. a minimal set of generators for  $\mathfrak{m}$ ), then  $n = \dim R$  and the  $\{x_i\}$  form a regular sequence. See the remark after ??; the point is that  $R/(x_1, \dots, x_{i-1})$  is regular for each  $i$  (by the aforementioned corollary), and hence a domain, so  $x_i$  acts on it by a nonzerodivisor.

The next example easily shows that a Cohen-Macaulay ring need not be regular, or even a domain:

**Example 2.3 (Local artinian rings are Cohen-Macaulay)** Any local artinian ring, because the dimension is zero for an artinian ring.

**Example 2.4 (Cohen-Macaulayness and completion)** A finitely generated module  $M$  is Cohen-Macaulay if and only if its completion  $\hat{M}$  is; this follows from ?? 16.3.

Here is a slightly harder example.

**Example 2.5** A normal local domain  $(R, \mathfrak{m})$  of dimension 2 is Cohen-Macaulay. This is a special case of Serre's criterion for normality.

Here is an argument. If  $x \in \mathfrak{m}$  is nonzero, we want to show that  $\text{depth } R/(x) = 1$ . To do this, we need to show that  $\mathfrak{m} \notin \text{Ass}(R/(x))$  for each such  $x$ , because then  $\text{depth } R/(x) \geq 1$  (which is all we need). However, suppose the contrary; then there is  $y$  not divisible by  $x$  such that  $\mathfrak{m}y \subset (x)$ . So  $y/x \notin R$ , but  $\mathfrak{m}(y/x) \subset R$ .

This, however, implies  $\mathfrak{m}$  is principal. Indeed, we either have  $\mathfrak{m}(y/x) = R$ , in which case  $\mathfrak{m}$  is generated by  $x/y$ , or  $\mathfrak{m}(y/x) \subset \mathfrak{m}$ . The latter would imply that  $y/x$  is integral over  $R$  (as multiplication by it stabilizes a finitely generated  $R$ -module), and by normality  $y/x \in R$ . We have seen much of this argument before.

**Example 2.6** Consider  $\mathbb{C}[x, y]/(xy)$ , the coordinate ring of the union of two axes intersecting at the origin. This is not regular, as its localization at the origin is not a domain. We will later show that this is a Cohen-Macaulay ring, though.

**Example 2.7**  $R = \mathbb{C}[x, y, z]/(xy, xz)$  is not Cohen-Macaulay (at the origin). The associated variety looks geometrically like the union of the plane  $x = 0$  and the line  $y = z = 0$  in affine 3-space. Here there are two components of different dimensions intersecting. Let's choose a regular sequence (that is, regular after localization at the origin). The dimension at the origin is clearly two because of the plane. First, we need a nonzerodivisor in this ring, which vanishes at the origin, say  $x + y + z$ . (Check this.) When we quotient by this, we get

$$S = \mathbb{C}[x, y, z]/(xy, xz, x + y + z) = \mathbb{C}[y, z]/((y + z)y, (y + z)z).$$

The claim is that  $S$  localized at the ideal corresponding to  $(0, 0)$  has depth zero. We have  $y + z \neq 0$ , which is killed by both  $y, z$ , and hence by the maximal ideal at zero. In particular the maximal ideal at zero is an associated prime, which implies the claim about the depth.

As it happens, a Cohen-Macaulay variety is always equidimensional. The rough reason is that each irreducible piece puts an upper bound on the depth given by the dimension of the piece. If any piece is too small, the total depth will be too small.

Here is the deeper statement:

**Proposition 2.8** *Let  $(R, \mathfrak{m})$  be a noetherian local ring,  $M$  a finitely generated, Cohen-Macaulay  $R$ -module. Then:*

1. *For each  $\mathfrak{p} \in \text{Ass}(M)$ , we have  $\dim M = \dim R/\mathfrak{p}$ .*
2. *Every associated prime of  $M$  is minimal (i.e. minimal in  $\text{supp } M$ ).*
3.  *$\text{supp } M$  is equidimensional.*

In general, there may be nontrivial inclusion relations among the associated primes of a general module. However, this cannot happen for a Cohen-Macaulay module.

*Proof.* The first statement implies all the others. (Recall that *equidimensional* means that all the irreducible components of  $\text{supp } M$ , i.e. the  $\text{Spec } R/\mathfrak{p}$ , have the same dimension.) But this in turn follows from the bound of Proposition 1.29. ▲

Next, we would like to obtain a criterion for when a quotient of a Cohen-Macaulay module is still Cohen-Macaulay. The answer will be similar to ?? for regular local rings.

**Proposition 2.9** *Let  $M$  be a Cohen-Macaulay module over the local noetherian ring  $(R, \mathfrak{m})$ . If  $x_1, \dots, x_n \in \mathfrak{m}$  is a  $M$ -regular sequence, then  $M/(x_1, \dots, x_n)M$  is Cohen-Macaulay of dimension (and depth)  $\dim M - n$ .*

*Proof.* Indeed, we reduce to the case  $n = 1$  by induction. But then, because  $x_1$  is a nonzerodivisor on  $M$ , we have  $\dim M/x_1M = \dim M - 1$  and  $\text{depth } M/x_1M = \text{depth } M - 1$ . Thus

$$\dim M/x_1M = \text{depth } M/x_1M. \quad \text{▲}$$

So, if we are given a Cohen-Macaulay module  $M$  and want one of a smaller dimension, we just have to find  $x \in \mathfrak{m}$  not contained in any of the minimal primes of  $\text{supp } M$  (these are the only associated primes). Then,  $M/xM$  will do the job.

## 2.2 The non-local case

More generally, we would like to make the definition:

**Definition 2.10** A general noetherian ring  $R$  is **Cohen-Macaulay** if  $R_{\mathfrak{p}}$  is Cohen-Macaulay for all  $\mathfrak{p} \in \text{Spec } R$ .

We should check that these definitions coincide for a local noetherian ring. This, however, is not entirely obvious; we have to show that localization preserves Cohen-Macaulayness. In this subsection, we shall do that, and we shall furthermore show that Cohen-Macaulay rings are *catenary*, or more generally that Cohen-Macaulay modules are catenary. (So far we have seen that they are equidimensional, in the local case.)

We shall deduce this from the following result, which states that for a Cohen-Macaulay module, we can choose partial systems of parameters in any given prime ideal in the support.

**Proposition 2.11** *Let  $M$  be a Cohen-Macaulay module over the local noetherian ring  $(R, \mathfrak{m})$ , and let  $\mathfrak{p} \in \text{supp } M$ . Let  $x_1, \dots, x_r \in \mathfrak{p}$  be a maximal  $M$ -sequence contained in  $\mathfrak{p}$ . Then:*

1.  $\mathfrak{p}$  is an associated and minimal prime of  $M/(x_1, \dots, x_r)M$ .
2.  $\dim R/\mathfrak{p} = \dim M - r$

*Proof.* We know (Proposition 2.9) that  $M/(x_1, \dots, x_r)M$  is a Cohen-Macaulay module too. Clearly  $\mathfrak{p}$  is in its support, since all the  $x_i \in \mathfrak{p}$ . The claim is that  $\mathfrak{p}$  is an associated prime—or minimal prime, it is the same thing—of  $M/(x_1, \dots, x_r)M$ . If not, there is  $x \in \mathfrak{p}$  that is a nonzerodivisor on this quotient, which means that  $\{x_1, \dots, x_r\}$  was not maximal as claimed.

Now we need to verify the assertion on the dimension. Clearly  $\dim M/(x_1, \dots, x_r)M = \dim M - r$ , and moreover  $\dim R/\mathfrak{p} = \dim M/(x_1, \dots, x_r)$  by Proposition 2.8. Combining these gives the second assertion. ▲

**Corollary 2.12** *Hypotheses as above,  $\dim M_{\mathfrak{p}} = r = \dim M - \dim R/\mathfrak{p}$ . Moreover,  $M_{\mathfrak{p}}$  is a Cohen-Macaulay module over  $R_{\mathfrak{p}}$ .*

This result shows that Definition 2.10 is a reasonable definition.

*Proof.* Indeed, if we consider the conclusions of ??, we find that  $x_1, \dots, x_r$  becomes a system of parameters for  $M_{\mathfrak{p}}$ : we have that  $M_{\mathfrak{p}}/(x_1, \dots, x_r)M_{\mathfrak{p}}$  is an artinian  $R_{\mathfrak{p}}$ -module, while the sequence is also regular. The first claim follows, as does the second: any module with a system of parameters that is a regular sequence is Cohen-Macaulay. ▲

As a result, we can get the promised result that a Cohen-Macaulay ring is catenary.

**Proposition 2.13** *If  $M$  is Cohen-Macaulay over the local noetherian ring  $R$ , then  $\text{supp } M$  is a catenary space.*

In other words, if  $\mathfrak{p} \subset \mathfrak{q}$  are elements of  $\text{supp } M$ , then every maximal chain of prime ideals from  $\mathfrak{p}$  to  $\mathfrak{q}$  has the same length.

*Proof.* We will show that  $\dim R/\mathfrak{p} = \dim R/\mathfrak{q} + \dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ , a claim that suffices to establish catenariness. We will do this by using the dimension formulas computed earlier.

Namely, we know that  $M$  is catenary over  $R$ , so by Corollary 2.12

$$\dim_{R_{\mathfrak{q}}} M_{\mathfrak{q}} = \dim M - \dim R/\mathfrak{q}, \quad \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \dim M - \dim R/\mathfrak{p}.$$

Moreover,  $M_{\mathfrak{q}}$  is Cohen-Macaulay over  $R_{\mathfrak{q}}$ . As a result, we have (in view of the previous equation)

$$\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \dim_{R_{\mathfrak{q}}} M_{\mathfrak{q}} - \dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} = \dim M - \dim R/\mathfrak{q} - \dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}.$$

Combining, we find

$$\dim M - \dim R/\mathfrak{p} = \dim M - \dim R/\mathfrak{q} - \dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}},$$

which is what we wanted. ▲

It thus follows that any Cohen-Macaulay ring, and thus any *quotient* of a Cohen-Macaulay ring, is catenary. In particular, it follows any non-catenary local noetherian ring cannot be expressed as a quotient of a Cohen-Macaulay (e.g. regular) local ring.

It also follows immediately that if  $R$  is any regular (not necessarily local) ring, then  $R$  is catenary, and the same goes for any quotient of  $R$ . In particular, since a polynomial ring over a field is regular, we find:

**Proposition 2.14** *Any affine ring is catenary.*

### 2.3 Reformulation of Serre's criterion

Much earlier, we proved criteria for a noetherian ring to be reduced and (more interestingly) normal. We can state them more cleanly using the theory of depth developed.

**Definition 2.15** Let  $R$  be a noetherian ring, and let  $k \in \mathbb{Z}_{\geq 0}$ .

1. We say that  $R$  satisfies **condition**  $R_k$  if, for every prime ideal  $\mathfrak{p} \in \text{Spec } R$  with  $\dim R_{\mathfrak{p}} \leq k$ , the local ring  $R_{\mathfrak{p}}$  is regular.
2.  $R$  satisfies **condition**  $S_k$  if  $\text{depth } R_{\mathfrak{p}} \geq \inf(k, \dim R_{\mathfrak{p}})$  for all  $\mathfrak{p} \in \text{Spec } R$ .

A Cohen-Macaulay ring satisfies all the conditions  $S_k$ , and conversely. The condition  $R_k$  means geometrically that the associated variety is regular (i.e., smooth, at least if one works over an algebraically closed field) outside a subvariety of codimension  $\geq k$ .

Recall that, according to ??, a noetherian ring is *reduced* iff:

1. For any minimal prime  $\mathfrak{p} \subset R$ ,  $R_{\mathfrak{p}}$  is a field.
2. Every associated prime of  $R$  is minimal.

Condition 1 can be restated as follows. The ideal  $\mathfrak{p} \subset R$  is minimal if and only if it is zero-dimensional, and  $R_{\mathfrak{p}}$  is regular if and only if it is a field. So the first condition is that *for every height zero prime,  $R_{\mathfrak{p}}$  is regular*. In other words, it is the condition  $R_0$ .

For the second condition,  $\mathfrak{p} \in \text{Ass}(R)$  iff  $\mathfrak{p} \in \text{Ass}(R_{\mathfrak{p}})$ , which is equivalent to  $\text{depth } R_{\mathfrak{p}} = 0$ . So the second condition states that for primes  $\mathfrak{p} \in \text{Spec } R$  of height at least 1,  $\mathfrak{p} \notin \text{Ass}(R_{\mathfrak{p}})$ , or  $\text{depth}(R_{\mathfrak{p}}) \geq 1$ . This is the condition  $S_1$ .

We find:

**Proposition 2.16** *A noetherian ring is reduced if and only if it satisfies  $R_0$  and  $S_1$ .*

In particular, for a Cohen-Macaulay ring, checking if it is reduced is easy; one just has to check  $R_0$  (if the localizations at minimal primes are reduced).

Serre's criterion for normality is in the same spirit, but harder. Recall that a noetherian ring is *normal* if it is a finite direct product of integrally closed domains.

The earlier form of Serre's criterion (see ??) was:

**Proposition 2.17** *Let  $R$  be a local ring. Then  $R$  is normal iff*

1.  $R$  is reduced.



2. For every height one prime  $\mathfrak{p} \in \text{Spec } R$ ,  $R_{\mathfrak{p}}$  is a DVR (i.e. regular).
3. For every nonzerodivisor  $x \in R$ , every associated prime of  $R/(x)$  is minimal.

In view of the criterion for reducedness, these conditions are equivalent to:

1. For every prime  $\mathfrak{p}$  of height  $\leq 1$ ,  $R_{\mathfrak{p}}$  is regular.
2. For every prime  $\mathfrak{p}$  of height  $\geq 1$ ,  $\text{depth } R_{\mathfrak{p}} \geq 1$  (necessary for reducedness)
3.  $\text{depth } R_{\mathfrak{p}} \geq 2$  for  $\mathfrak{p}$  containing but not minimal over any principal ideal  $(x)$  for  $x$  a nonzerodivisor. This is the last condition of the proposition; to say  $\text{depth } R_{\mathfrak{p}} \geq 2$  is to say that  $\text{depth } R_{\mathfrak{p}}/(x)R_{\mathfrak{p}} \geq 1$ , or  $\mathfrak{p} \notin \text{Ass}(R_{\mathfrak{p}}/(x)R_{\mathfrak{p}})$ .

Combining all this, we find:

**Theorem 2.18 (Serre's criterion)** *A noetherian ring is normal if and only if it satisfies the conditions  $R_1$  and  $S_2$ .*

Again, for a Cohen-Macaulay ring, the last condition is automatic, as the depth is the codimension.

### §3 Projective dimension and free resolutions

We shall introduce the notion of *projective dimension* of a module; this will be the smallest projective resolution it admits (if there is none such, the dimension is  $\infty$ ). We can think of it as measuring how far a module is from being projective. Over a noetherian *local* ring, we will show that the projective dimension can be calculated very simply using the Tor functor (which is an elaboration of the story that a projective module over a local ring is free).

Ultimately we want to show that a noetherian local ring is regular if and only if every finitely generated module admits a finite free resolution. Although we shall not get to that result until the next section, we will at least relate projective dimension to a more familiar invariant of a module: *depth*.

#### 3.1 Introduction

Let  $R$  be a commutative ring,  $M$  an  $R$ -module.

**Definition 3.1** The **projective dimension** of  $M$  is the largest integer  $n$  such that there exists a module  $N$  with

$$\text{Ext}^n(M, N) \neq 0.$$

We allow  $\infty$ , if arbitrarily large such  $n$  exist. We write  $\text{pd}(M)$  for the projective dimension. For convenience, we set  $\text{pd}(0) = -\infty$ .

So, if  $m > n = \text{pd}(M)$ , then we have  $\text{Ext}^m(M, N) = 0$  for *all* modules  $N$ , and  $n$  is the smallest integer with this property. As an example, note that  $\text{pd}(M) = 0$  if and only if  $M$  is projective and nonzero. Indeed, we have seen that the Ext groups  $\text{Ext}^i(M, N)$ ,  $i > 0$  vanish always for  $M$  projective, and conversely.

To compute  $\text{pd}(M)$  in general, one can proceed as follows. Take any  $M$ . Choose a surjection  $P \rightarrow M$  with  $P$  projective; call the kernel  $K$  and draw a short exact sequence

$$0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0.$$

For any  $R$ -module  $N$ , we have a long exact sequence

$$\text{Ext}^{i-1}(P, N) \rightarrow \text{Ext}^{i-1}(K, N) \rightarrow \text{Ext}^i(M, N) \rightarrow \text{Ext}^i(P, N).$$

If  $i > 0$ , the right end vanishes; if  $i > 1$ , the left end vanishes. So if  $i > 1$ , this map  $\text{Ext}^{i-1}(K, N) \rightarrow \text{Ext}^i(M, N)$  is an *isomorphism*.

Suppose that  $\text{pd}(K) = d \geq 0$ . We find that  $\text{Ext}^{i-1}(K, N) = 0$  for  $i - 1 > d$ . This implies that  $\text{Ext}^i(M, N) = 0$  for such  $i > d + 1$ . In particular,  $\text{pd}(M) \leq d + 1$ . This argument is completely reversible if  $d > 0$ . Then we see from these isomorphisms that

$$\boxed{\text{pd}(M) = \text{pd}(K) + 1}, \quad \text{unless } \text{pd}(M) = 0 \tag{16.3}$$

If  $M$  is projective, the sequence  $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$  splits, and  $\text{pd}(K) = 0$  too.

The upshot is that we can compute projective dimension by choosing a projective resolution.

**Proposition 3.2** *Let  $M$  be an  $R$ -module. Then  $\text{pd}(M) \leq n$  iff there exists a finite projective resolution of  $M$  having  $n + 1$  terms,*

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

*Proof.* Induction on  $n$ . When  $n = 0$ ,  $M$  is projective, and we can use the resolution  $0 \rightarrow M \rightarrow M \rightarrow 0$ .

Suppose  $\text{pd}(M) \leq n$ , where  $n > 0$ . We can get a short exact sequence

$$0 \rightarrow K \rightarrow P_0 \rightarrow M \rightarrow 0$$

with  $P_0$  projective, so  $\text{pd}(K) \leq n - 1$  by (16.3). The inductive hypothesis implies that there is a projective resolution of  $K$  of length  $\leq n - 1$ . We can splice this in with the short exact sequence to get a projective resolution of  $M$  of length  $n$ .

The argument is reversible. Choose any projective resolution

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

and split into short exact sequences, and then one argue inductively to show that  $\text{pd}(M) \leq n$ .  $\blacktriangle$

Let  $\text{pd}(M) = n$ . Choose any projective resolution  $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M$ . Choose  $K_i = \ker(P_i \rightarrow P_{i-1})$  for each  $i$ . Then there is a short exact sequence  $0 \rightarrow K_0 \rightarrow P_0 \rightarrow M \rightarrow 0$ . Moreover, there are exact sequences

$$0 \rightarrow K_i \rightarrow P_i \rightarrow K_{i-1} \rightarrow 0$$

for each  $i$ . From these, and from (16.3), we see that the projective dimensions of the  $K_i$  drop by one as  $i$  increments. So  $K_{n-1}$  is projective if  $\text{pd}(M) = n$  as  $\text{pd}(K_{n-1}) = 0$ . In particular, we can get a projective resolution

$$0 \rightarrow K_{n-1} \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

which is of length  $n$ . In particular, if one has a (possibly infinite) projective resolution  $M$ , one can stop after going out  $n$  terms, because the kernels will become projective. In other words, the projective resolution can be made to *break off* at the  $n$ th term. This applies to *any* projective resolution. Conversely, since any module has a (possibly infinite) projective resolution, we find:

**Proposition 3.3** *We have  $\text{pd}(M) \leq n$  if any projective resolution*

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

*breaks off at the  $n$ th stage: that is, the kernel of  $P_{n-1} \rightarrow P_{n-2}$  is projective.*

If  $\text{pd}(M) \leq n$ , then by definition we have  $\text{Ext}^{n+1}(M, N) = 0$  for *any* module  $N$ . By itself, this does not say anything about the Tor functors. However, the criterion for projective dimension enables us to show:

**Proposition 3.4** *If  $\text{pd}(M) \leq n$ , then  $\text{Tor}_m(M, N) = 0$  for  $m > n$ .*

One can define an analog of projective dimension with the Tor functors, called *flat dimension*, and it follows that the flat dimension is at most the projective dimension.

In fact, we have more generally:

**Proposition 3.5** *Let  $F$  be a right-exact functor on the category of  $R$ -modules, and let  $\{L_i F\}$  be its left derived functors. If  $\text{pd}(M) \leq n$ , then  $L_i F(M) = 0$  for  $i > n$ .*

Clearly this implies the claim about Tor functors.

*Proof.* Recall how  $L_i F(M)$  can be computed. Namely, one chooses a projective resolution  $P_\bullet \rightarrow M$  (any will do), and compute the homology of the complex  $F(P_\bullet)$ . However, we can choose  $P_\bullet \rightarrow M$  such that  $P_i = 0$  for  $i > n$  by Proposition 3.2. Thus  $F(P_\bullet)$  is concentrated in degrees between 0 and  $n$ , and the result becomes clear when one takes the homology.  $\blacktriangle$

In general, flat modules are not projective (e.g.  $\mathbb{Q}$  is flat, but not projective, over  $\mathbb{Z}$ ), and while one can use projective dimension to bound “flat dimension” (the analog for Tor-vanishing), one cannot use the flat dimension to bound the projective dimension. For a local ring, we will see that it is possible in the next subsection.

### 3.2 Tor and projective dimension

Over a noetherian *local* ring, there is a much simpler way to test whether a finitely generated module is projective. This is a special case of the very general flatness criterion ??, but we can give a simple direct proof. So we prefer to keep things self-contained.

**Theorem 3.6** *Let  $M$  be a finitely generated module over the noetherian local ring  $(R, \mathfrak{m})$ , with residue field  $k = R/\mathfrak{m}$ . Then, if  $\text{Tor}_1(M, k) = 0$ ,  $M$  is free.*

In particular, projective—or even flat—modules which are of finite type over  $R$  are automatically free. This is a strengthening of the earlier theorem (??) that a finitely generated projective module over a local ring is free.

*Proof.* Indeed, we can find a free module  $F$  and a surjection  $F \rightarrow M$  such that  $F \otimes_R k \rightarrow M \otimes_R k$  is an isomorphism. To do this, choose elements of  $M$  that form a basis of  $M \otimes_R k$ , and then define a map  $F \rightarrow M$  via these elements; it is a surjection by Nakayama’s lemma.

Let  $K$  be the kernel of  $F \rightarrow M$ , so there is an exact sequence

$$0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0.$$

We want to show that  $K = 0$ , which will imply that  $M = 0$ . By Nakayama’s lemma, it suffices to show that  $K \otimes_R k = 0$ . But we have an exact sequence

$$\text{Tor}_1(M, k) \rightarrow K \otimes_R k \rightarrow F \otimes_R k \rightarrow M \otimes_R k \rightarrow 0.$$

The last map is an isomorphism, and  $\text{Tor}_1(M, k) = 0$ , which implies that  $K \otimes_R k = 0$ . The result is now proved.  $\blacktriangle$

As a result, we can compute the projective dimension of a module in terms of Tor.

**Corollary 3.7** *Let  $M$  be a finitely generated module over the noetherian local ring  $R$  with residue field  $k$ . Then  $\text{pd}(M)$  is the largest integer  $n$  such that  $\text{Tor}_n(M, k) \neq 0$ . It is also the smallest integer  $n$  such that  $\text{Tor}_{n+1}(M, k) = 0$ .*

There is a certain symmetry: if  $\text{Ext}$  replaces  $\text{Tor}$ , then one has the definition of depth. We will show later that there is indeed a useful connection between projective dimension and depth.

*Proof.* We will show that if  $\text{Tor}_{n+1}(M, k) = 0$ , then  $\text{pd}(M) \leq n$ . This implies the claim, in view of Proposition 3.4. Choose a (possibly infinite) projective resolution

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

Since  $R$  is noetherian, we can assume that each  $P_i$  is *finitely generated*.

Write  $K_i = \ker(P_i \rightarrow P_{i-1})$ , as before; these are finitely generated  $R$ -modules. We want to show that  $K_{n-1}$  is projective, which will establish the claim, as then the projective resolution will “break off.” But we have an exact sequence

$$0 \rightarrow K_0 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

which shows that  $\text{Tor}_n(K_0, k) = \text{Tor}_{n+1}(M, k) = 0$ . Using the exact sequence  $0 \rightarrow K_i \rightarrow P_i \rightarrow K_{i-1} \rightarrow 0$ , we inductively work downwards to get that  $\text{Tor}_1(K_{n-1}, k) = 0$ . So  $K_{n-1}$  is projective by Theorem 3.6.  $\blacktriangle$

In particular, we find that if  $\text{pd}(k) \leq n$ , then  $\text{pd}(M) \leq n$  for all  $M$ . This is because if  $\text{pd}(k) \leq n$ , then  $\text{Tor}_{n+1}(M, k) = 0$  by using the relevant resolution of  $k$  (see Proposition 3.4, but for  $k$ ).

**Corollary 3.8** *Suppose there exists  $n$  such that  $\text{Tor}_{n+1}(k, k) = 0$ . Then every finitely generated  $R$ -module has a finite free resolution of length at most  $n$ .*

We have thus seen that  $k$  is in some sense the “worst”  $R$ -module, in that it is as far from being projective, or that it has the largest projective dimension. We can describe this worst-case behavior with the next concept:

**Definition 3.9** Given a ring  $R$ , the **global dimension** is the sup of the projective dimensions of all finitely generated  $R$ -modules.

So, to recapitulate: the global dimension of a noetherian local ring  $R$  is the projective dimension of its residue field  $k$ , or even the *flat* dimension of the residue field.

### 3.3 Minimal projective resolutions

Usually projective resolutions are non-unique; they are only unique up to chain homotopy. We will introduce a certain restriction that enforces uniqueness. These “minimal” projective resolutions will make it extremely easy to compute the groups  $\text{Tor}_\bullet(\cdot, k)$ .

Let  $(R, \mathfrak{m})$  be a local noetherian ring with residue field  $k$ ,  $M$  a finitely generated  $R$ -module. All tensor products will be over  $R$ .

**Definition 3.10** A projective resolution  $P_\bullet \rightarrow M$  of finitely generated modules is **minimal** if for each  $i$ , the induced map  $P_i \otimes k \rightarrow P_{i-1} \otimes k$  is zero, and the map  $P_0 \otimes k \rightarrow M/\mathfrak{m}M$  is an isomorphism.

In other words, the complex  $P_\bullet \otimes k$  is isomorphic to  $M \otimes k$ . This is equivalent to saying that for each  $i$ , the map  $P_i \rightarrow \ker(P_{i-1} \rightarrow P_{i-2})$  is an isomorphism modulo  $\mathfrak{m}$ .

**Proposition 3.11** *Every  $M$  (over a local noetherian ring) has a minimal projective resolution.*

*Proof.* Start with a module  $M$ . Then  $M/\mathfrak{m}M$  is a finite-dimensional vector space over  $k$ , of dimension say  $d_0$ . We can choose a basis for that vector space, which we can lift to  $M$ . That determines a map of free modules

$$R^{d_0} \rightarrow M,$$

which is a surjection by Nakayama's lemma. It is by construction an isomorphism modulo  $\mathfrak{m}$ . Then define  $K = \ker(R^{d_0} \rightarrow M)$ ; this is finitely generated by noetherianness, and we can do the same thing for  $K$ , and repeat to get a map  $R^{d_1} \rightarrow K$  which is an isomorphism modulo  $\mathfrak{m}$ . Then

$$R^{d_1} \rightarrow R^{d_0} \rightarrow M \rightarrow 0$$

is exact, and minimal; we can continue this by the same procedure. ▲

**Proposition 3.12** *Minimal projective resolutions are unique up to isomorphism.*

*Proof.* Suppose we have one minimal projective resolution:

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

and another:

$$\cdots \rightarrow Q_2 \rightarrow Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0.$$

There is always a map of projective resolutions  $P_* \rightarrow Q_*$  by general homological algebra. There is, equivalently, a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \text{id} & \\ \cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

If both resolutions are minimal, the claim is that this map is an isomorphism. That is,  $\phi_i : P_i \rightarrow Q_i$  is an isomorphism, for each  $i$ .

To see this, note that  $P_i, Q_i$  are finite free  $R$ -modules.<sup>1</sup> So  $\phi_i$  is an isomorphism iff  $\phi_i$  is an isomorphism modulo the maximal ideal, i.e. if

$$P_i/\mathfrak{m}P_i \rightarrow Q_i/\mathfrak{m}Q_i$$

is an isomorphism. Indeed, if  $\phi_i$  is an isomorphism, then its tensor product with  $R/\mathfrak{m}$  obviously is an isomorphism. Conversely suppose that the reductions mod  $\mathfrak{m}$  make an isomorphism. Then the ranks of  $P_i, Q_i$  are the same, and  $\phi_i$  is an  $n$ -by- $n$  matrix whose determinant is not in the maximal ideal, so is invertible. This means that  $\phi_i$  is invertible by the usual formula for the inverse matrix.

So we are to check that  $P_i/\mathfrak{m}P_i \rightarrow Q_i/\mathfrak{m}Q_i$  is an isomorphism for each  $i$ . This is equivalent to the assertion that

$$(Q_i/\mathfrak{m}Q_i)^\vee \rightarrow (P_i/\mathfrak{m}P_i)^\vee$$

is an isomorphism. But this is the map

$$\mathrm{Hom}_R(Q_i, R/\mathfrak{m}) \rightarrow \mathrm{Hom}_R(P_i, R/\mathfrak{m}).$$

If we look at the chain complexes  $\mathrm{Hom}(P_*, R/\mathfrak{m}), \mathrm{Hom}(Q_*, R/\mathfrak{m})$ , the cohomologies compute the Ext groups of  $(M, R/\mathfrak{m})$ . But all the maps in this chain complex are zero because the resolution is minimal, and we have that the image of  $P_i$  is contained in  $\mathfrak{m}P_{i-1}$  (ditto for  $Q_i$ ). So the cohomologies are just the individual terms, and the maps  $\mathrm{Hom}_R(Q_i, R/\mathfrak{m}) \rightarrow \mathrm{Hom}_R(P_i, R/\mathfrak{m})$  correspond to the identities on  $\mathrm{Ext}^i(M, R/\mathfrak{m})$ . So these are isomorphisms.<sup>2</sup> ▲

<sup>1</sup>We are using the fact that a finite projective module over a local ring is *free*.

<sup>2</sup>We are sweeping under the rug the statement that Ext can be computed via *any* projective resolution. More precisely, if you take any two projective resolutions, and take the induced maps between the projective resolutions, hom them into  $R/\mathfrak{m}$ , then the maps on cohomology are isomorphisms.

**Corollary 3.13** *If  $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M$  is a minimal projective resolution of  $M$ , then the ranks  $\text{rank}(P_i)$  are well-defined (i.e. don't depend on the choice of the minimal resolution).*

*Proof.* Immediate from the proposition. In fact, the ranks are the dimensions (as  $R/\mathfrak{m}$ -vector spaces) of  $\text{Ext}^i(M, R/\mathfrak{m})$ . ▲

### 3.4 The Auslander-Buchsbaum formula

**Theorem 3.14 (Auslander-Buchsbaum formula)** *Let  $R$  be a local noetherian ring,  $M$  a finitely generated  $R$ -module of finite projective dimension. If  $\text{pd}(R) < \infty$ , then  $\text{pd}(M) = \text{depth}(R) - \text{depth}(M)$ .*

*Proof.* Induction on  $\text{pd}(M)$ . When  $\text{pd}(M) = 0$ , then  $M$  is projective, so isomorphic to  $R^n$  for some  $n$ . Thus  $\text{depth}(M) = \text{depth}(R)$ .

Assume  $\text{pd}(M) > 0$ . Choose a surjection  $P \twoheadrightarrow M$  and write an exact sequence

$$0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0,$$

where  $\text{pd}(K) = \text{pd}(M) - 1$ . We also know by induction that

$$\text{pd}(K) = \text{depth } R - \text{depth}(K).$$

What we want to prove is that

$$\text{depth } R - \text{depth } M = \text{pd}(M) = \text{pd}(K) + 1.$$

This is equivalent to wanting know that  $\text{depth}(K) = \text{depth}(M) + 1$ . In general, this may not be true, though, but we will prove it under minimality hypotheses.

Without loss of generality, we can choose that  $P$  is *minimal*, i.e. becomes an isomorphism modulo the maximal ideal  $\mathfrak{m}$ . This means that the rank of  $P$  is  $\dim M/\mathfrak{m}M$ . So  $K = 0$  iff  $P \rightarrow M$  is an isomorphism; we've assumed that  $M$  is not free, so  $K \neq 0$ .

Recall that the depth of  $M$  is the smallest value  $i$  such that  $\text{Ext}^i(R/\mathfrak{m}, M) \neq 0$ . So we should look at the long exact sequence from the above short exact sequence:

$$\text{Ext}^i(R/\mathfrak{m}, P) \rightarrow \text{Ext}^i(R/\mathfrak{m}, M) \rightarrow \text{Ext}^{i+1}(R/\mathfrak{m}, K) \rightarrow \text{Ext}^{i+1}(R/\mathfrak{m}, P).$$

Now  $P$  is just a direct sum of copies of  $R$ , so  $\text{Ext}^i(R/\mathfrak{m}, P)$  and  $\text{Ext}^{i+1}(R/\mathfrak{m}, P)$  are zero if  $i + 1 < \text{depth } R$ . In particular, if  $i + 1 < \text{depth } R$ , then the map  $\text{Ext}^i(R/\mathfrak{m}, M) \rightarrow \text{Ext}^{i+1}(R/\mathfrak{m}, K)$  is an isomorphism. So we find that  $\text{depth } M + 1 = \text{depth } K$  in this case.

We have seen that *if*  $\text{depth } K < \text{depth } R$ , *then* by taking  $i$  over all integers  $< \text{depth } K$ , we find that

$$\text{Ext}^i(R/\mathfrak{m}, M) = \begin{cases} 0 & \text{if } i + 1 < \text{depth } K \\ \text{Ext}^{i+1}(R/\mathfrak{m}, K) & \text{if } i + 1 = \text{depth } K \end{cases}.$$

In particular, we are **done** unless  $\text{depth } K \geq \text{depth } R$ . By the inductive hypothesis, this is equivalent to saying that  $K$  is projective.

So let us consider the case where  $K$  is projective, i.e.  $\text{pd}(M) = 1$ . We want to show that  $\text{depth } M = d - 1$  if  $d = \text{depth } R$ . We need a slightly different argument in this case. Let  $d = \text{depth}(R) = \text{depth}(P) = \text{depth}(K)$  since  $P, K$  are free. We have a short exact sequence

$$0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$$

and a long exact sequence of Ext groups:

$$0 \rightarrow \text{Ext}^{d-1}(R/\mathfrak{m}, M) \rightarrow \text{Ext}^d(R/\mathfrak{m}, K) \rightarrow \text{Ext}^d(R/\mathfrak{m}, P).$$

We know that  $\text{Ext}^d(R/\mathfrak{m}, K)$  is nonzero as  $K$  is free and  $R$  has depth  $d$ . However,  $\text{Ext}^i(R/\mathfrak{m}, K) = \text{Ext}^i(R/\mathfrak{m}, P) = 0$  for  $i < d$ . This implies that  $\text{Ext}^{i-1}(R/\mathfrak{m}, M) = 0$  for  $i < d$ .

We will show:

The map  $\text{Ext}^d(R/\mathfrak{m}, K) \rightarrow \text{Ext}^d(R/\mathfrak{m}, P)$  is zero.

This will imply that the depth of  $M$  is *precisely*  $d - 1$ . This is because the matrix  $K \rightarrow P$  is given by multiplication by a matrix with coefficients in  $\mathfrak{m}$  as  $K/\mathfrak{m}K \rightarrow P/\mathfrak{m}P$  is zero. In particular, the map on the Ext groups is zero, because it is annihilated by  $\mathfrak{m}$ .  $\blacktriangle$

**Example 3.15** Consider the case of a *regular* local ring  $R$  of dimension  $n$ . Then  $\text{depth}(R) = n$ , so we have

$$\text{pd}(M) + \text{depth}(M) = n,$$

for every finitely generated  $R$ -module  $M$ . In particular,  $\text{depth}(M) = n$  if and only if  $M$  is free.

**Example 3.16 (The Cohen-Macaulay locus is open)** Let  $R$  be a regular noetherian ring (i.e. one all of whose localizations are regular). Let  $M$  be a finitely generated  $R$ -module. We consider the locus  $Z \subset \text{Spec } R$  consisting of prime ideals  $\mathfrak{p} \in \text{Spec } R$  such that  $M_{\mathfrak{p}}$  is a Cohen-Macaulay  $R$ -module. We want to show that this is an *open* subset.

Namely, over a local ring  $(A, \mathfrak{m})$ , define the *codepth* of a finitely generated  $A$ -module  $N$  as  $\text{codepth} N = \dim N - \text{depth} N \geq 0$ ; we have that  $\text{codepth} N = 0$  if and only if  $N$  is Cohen-Macaulay. We are going to show that the function  $\mathfrak{p} \mapsto \text{codepth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$  is upper semicontinuous on  $\text{Spec } R$ . To do this, we use the Auslander-Buchsbaum formula  $\text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \dim R_{\mathfrak{p}} - \text{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$  (see Example 3.15). We will show below that  $\mathfrak{p} \mapsto \text{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$  is upper semi-continuous on  $\text{Spec } R$ . Thus, we have

$$\text{codepth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = -(\dim R_{\mathfrak{p}} - \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}}) + \text{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}},$$

where the second term is upper semi-continuous. The claim is that the first term is upper semi-continuous. If we consider  $\text{supp } M \subset \text{Spec } R$ , then the bracketed difference measures the *local codimension* of  $\text{supp } M \subset \text{Spec } R$ . Namely,  $\dim R_{\mathfrak{p}} - \dim \text{supp } M_{\mathfrak{p}}$  is the local codimension because  $R_{\mathfrak{p}}$  is regular, and consequently  $\text{Spec } R_{\mathfrak{p}}$  is biequidimensional (**TO BE ADDED:** argument). The local codimension of any set is always lower semi-continuous (**TO BE ADDED:** reference in the section on topological dim). As a result, the codepth is upper semi-continuous.

We just need to prove the assertion that  $\mathfrak{p} \mapsto \text{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$  is upper semi-continuous. That is, we need to show that if  $M_{\mathfrak{p}}$  admits a projective resolution of length  $n$  by finitely generated modules, then there is a projective resolution of length  $n$  of  $M_{\mathfrak{q}}$  for  $\mathfrak{q}$  in some Zariski neighborhood. But a projective resolution of  $M_{\mathfrak{p}}$  “descends” to a projective (even free) resolution of  $M_g$  for some  $g \notin \mathfrak{p}$ , which gives the result by localization.

If  $R$  is the *quotient* of a regular ring, the same result holds (because the Cohen-Macaulay locus behaves properly with respect to quotients). In particular, this result holds for  $R$  an affine ring.

**Example 3.17** Let  $R = \mathbb{C}[x_1, \dots, x_n]/\mathfrak{p}$  for  $\mathfrak{p}$  prime. Choose an injection  $R' \rightarrow R$  where  $R' = \mathbb{C}[y_1, \dots, y_m]$  and  $R$  is a finitely generated  $R'$ -module. This exists by the Noether normalization lemma.

We wanted to show:

**Theorem 3.18**  $R$  is Cohen-Macaulay<sup>3</sup> iff  $R$  is a projective  $R'$ -module.

We shall use the fact that projectiveness can be tested locally at every maximal ideal.

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<sup>3</sup>That is, its localizations at any prime—or, though we haven’t proved yet, at any maximal ideal—are.

*Proof.* Choose a maximal ideal  $\mathfrak{m} \subset R'$ . We will show that  $R_{\mathfrak{m}}$  is a free  $R'_{\mathfrak{m}}$ -module via the injection of rings  $R'_{\mathfrak{m}} \hookrightarrow R_{\mathfrak{m}}$  (where  $R_{\mathfrak{m}}$  is defined as  $R$  localized at the multiplicative subset of elements of  $R' - \mathfrak{m}$ ) at each  $\mathfrak{m}$  iff Cohen-Macaulayness holds.

Now  $R'_{\mathfrak{m}}$  is a regular local ring, so its depth is  $m$ . By the Auslander-Buchsbaum formula,  $R_{\mathfrak{m}}$  is projective as an  $R'_{\mathfrak{m}}$ -module iff

$$\text{depth}_{R'_{\mathfrak{m}}} R_{\mathfrak{m}} = m.$$

Now  $R$  is a projective module iff the above condition holds for all maximal ideals  $\mathfrak{m} \subset R'$ . The claim is that this is equivalent to saying that  $\text{depth } R_{\mathfrak{n}} = m = \dim R_{\mathfrak{n}}$  for every maximal ideal  $\mathfrak{n} \subset R$  (depth over  $R$ !).

These two statements are almost the same, but one is about the depth of  $R$  as an  $R$ -module, and another as an  $R'$ -module.

Issue: There may be several maximal ideals of  $R$  lying over the maximal ideal  $\mathfrak{m} \subset R'$ .

The problem is that  $R_{\mathfrak{m}}$  is not generally local, and not generally equal to  $R_{\mathfrak{n}}$  if  $\mathfrak{n}$  lies over  $\mathfrak{m}$ . Fortunately, depth makes sense even over semi-local rings (rings with finitely many maximal ideals).

Let us just assume that this does not occur, though. Let us assume that  $R_{\mathfrak{m}}$  is a local ring for every maximal ideal  $\mathfrak{m} \subset R$ . Then we are reduced to showing that if  $S = R_{\mathfrak{m}}$ , then the depth of  $S$  as an  $R'_{\mathfrak{m}}$ -module is the same as the depth as an  $R_{\mathfrak{m}}$ -module. That is, the depth doesn't depend too much on the ring, since  $R'_{\mathfrak{m}}, R_{\mathfrak{m}}$  are "pretty close." If you believe this, then you believe the theorem, by the first paragraph.

Let's prove this claim in a more general form:

**Proposition 3.19** *Let  $\phi : S' \rightarrow S$  be a local<sup>4</sup> map of local noetherian rings such that  $S$  is a finitely generated  $S'$ -module. Then, for any finitely generated  $S$ -module  $M$ ,*

$$\text{depth}_S M = \text{depth}_{S'} M.$$

With this, the theorem will be proved.

**Remark** This result generalizes to the semi-local case, which is how one side-steps the issue above.

*Proof.* By induction on  $\text{depth}_{S'} M$ . There are two cases.

Let  $\mathfrak{m}', \mathfrak{m}$  be the maximal ideals of  $S', S$ . If  $\text{depth}_{S'}(M) > 0$ , then there is an element  $a$  in  $\mathfrak{m}'$  such that

$$M \xrightarrow{\phi(a)} M$$

is injective. Now  $\phi(a) \in \mathfrak{m}$ . So  $\phi(a)$  is a nonzerodivisor, and we have an exact sequence

$$0 \rightarrow M \xrightarrow{\phi(a)} M \rightarrow M/\phi(a)M \rightarrow 0.$$

Thus we find

$$\text{depth}_S M > 0.$$

Moreover, we find that  $\text{depth}_S M = \text{depth}_S(M/\phi(a)M) + 1$  and  $\text{depth}_{S'} M = \text{depth}_{S'}(M/\phi(a)M) + 1$ . The inductive hypothesis now tells us that

$$\text{depth}_S M = \text{depth}_{S'} M. \quad \blacktriangle$$

The hard case is where  $\text{depth}_{S'} M = 0$ . We need to show that this is equivalent to  $\text{depth}_S M = 0$ . So we know at first that  $\mathfrak{m}' \in \text{Ass}(M)$ . That is, there is an element  $x \in M$  such that  $\text{Ann}_{S'}(x) = \mathfrak{m}'$ . Now  $\text{Ann}_S(x) \subsetneq S$  and contains  $\mathfrak{m}'S$ .

---

<sup>4</sup>I.e.  $\phi$  sends non-units into non-units.



$Sx \subset M$  is a submodule, surjected onto by  $S$  by the map  $a \rightarrow ax$ . This map actually, as we have seen, factors through  $S/\mathfrak{m}'S$ . Here  $S$  is a finite  $S'$ -module, so  $S/\mathfrak{m}'S$  is a finite  $S'/\mathfrak{m}'$ -module. In particular, it is a finite-dimensional vector space over a field. It is thus a local artinian ring. But  $Sx$  is a module over this local artinian ring. It must have an associated prime, which is a maximal ideal in  $S/\mathfrak{m}'S$ . The only maximal ideal can be  $\mathfrak{m}/\mathfrak{m}'S$ . It follows that  $\mathfrak{m} \in \text{Ass}(Sx) \subset \text{Ass}(M)$ .

In particular,  $\text{depth}_S M = 0$  too, and we are done.  $\blacktriangle$

## §4 Serre's criterion and its consequences

We would like to prove Serre's criterion for regularity.

**Theorem 4.1** *Let  $(R, \mathfrak{m})$  be a local noetherian ring. Then  $R$  is regular iff  $R/\mathfrak{m}$  has finite projective dimension. In this case,  $\text{pd}(R/\mathfrak{m}) = \dim R$ .*

**TO BE ADDED:** proof

### 4.1 First consequences

**Proposition 4.2** *Let  $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  be a flat, local homomorphism of noetherian local rings. If  $S$  is regular, so is  $R$ .*

*Proof.* Let  $n = \dim S$ . Let  $M$  be a finitely generated  $R$ -module, and consider a resolution

$$P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where all the  $\{P_i\}$  are finite free  $R$ -modules. If we can show that the kernel of  $P_n \rightarrow P_{n-1}$  is projective, then it will follow that  $M$  has finite projective dimension. Since  $M$  was arbitrary, it will follow that  $R$  is regular too, by Serre's criterion.

Let  $K$  be the kernel, so there is an exact sequence

$$0 \rightarrow K \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0,$$

which we can tensor with  $S$ , by flatness:

$$0 \rightarrow K \otimes_R S \rightarrow P_n \otimes_R S \rightarrow P_{n-1} \otimes_R S \rightarrow \cdots \rightarrow P_0 \otimes_R S \rightarrow M \otimes_R S \rightarrow 0.$$

Because any finitely generated  $S$ -module has projective dimension  $\leq n$ , it follows that  $K \otimes_R S$  is projective, and in particular flat.

But now  $S$  is *faithfully flat* over  $R$  (see ??), and it follows that  $K$  is  $R$ -flat. Thus  $K$  is projective over  $R$ , proving the claim.  $\blacktriangle$

**Theorem 4.3** *The localization of a regular local ring at a prime ideal is regular.*

Geometrically, this means that to test whether a nice scheme (e.g. a variety) is regular (i.e., all the local rings are regular), one only has to test the *closed* points.

*Proof.* Let  $(R, \mathfrak{m})$  be a regular local ring. Let  $\mathfrak{p} \in \text{Spec } R$  be a prime ideal; we wish to show that  $R_{\mathfrak{p}}$  is regular. To do this, let  $M$  be a finitely generated  $R_{\mathfrak{p}}$ -module. Then we can find a finitely generated  $R$ -submodule  $N \subset M$  such that the natural map  $N_{\mathfrak{p}} \rightarrow M$  is an isomorphism. If we take a finite free resolution of  $N$  by  $R$ -modules and localize at  $\mathfrak{p}$ , we get a finite free resolution of  $M$  by  $R_{\mathfrak{p}}$ -modules.

It now follows that  $M$  has finite projective dimension as an  $R_{\mathfrak{p}}$ -module. By Serre's criterion, this implies that  $R_{\mathfrak{p}}$  is regular.  $\blacktriangle$

## 4.2 Regular local rings are factorial

We now aim to prove that a regular local ring is factorial.

First, we need:

**Definition 4.4** Let  $R$  be a noetherian ring and  $M$  a f.gen.  $R$ -module. Then  $M$  is **stably free** if  $M \oplus R^k$  is free for some  $k$ .

Stably free obviously implies “projective.” Free implies stably free, clearly—take  $k = 0$ . Over a local ring, a finitely generated projective module is free, so all three notions are equivalent. Over a general ring, these notions are generally different.

We will need the following lemma:

**Lemma 4.5** *Let  $M$  be an  $R$ -module with a finite free resolution. If  $M$  is projective, it is stably free.*

*Proof.* There is an exact sequence

$$0 \rightarrow F_k \rightarrow F_{k-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

with the  $F_i$  free and finitely generated, by assumption.

We induct on the length  $k$  of the resolution. We know that if  $N$  is the kernel of  $F_0 \rightarrow M$ , then  $N$  is projective (as the sequence  $0 \rightarrow N \rightarrow F_0 \rightarrow M \rightarrow 0$  splits) so there is a resolution

$$0 \rightarrow F_k \rightarrow \cdots \rightarrow F_1 \rightarrow N \rightarrow 0.$$

By the inductive hypothesis,  $N$  is stably free. So there is a free module  $R^d$  such that  $N \oplus R^d$  is free.

We know that  $M \oplus N = F_0$  is free. Thus  $M \oplus N \oplus R^d = F_0 \oplus R^d$  is free and  $N \oplus R^d$  is free. Thus  $M$  is stably free.  $\blacktriangle$

**Remark** Stably freeness does **not** generally imply freeness, though it does over a local noetherian ring.

Nonetheless,

**Proposition 4.6** *Stably free does imply free for invertible modules.*

*Proof.* Let  $I$  be stably free and invertible. We must show that  $I \simeq R$ . Without loss of generality, we can assume that  $\text{Spec } R$  is connected, i.e.  $R$  has no nontrivial idempotents. We will assume this in order to talk about the **rank** of a projective module.

We know that  $I \oplus R^n \simeq R^m$  for some  $m$ . We know that  $m = n + 1$  by localization. So  $I \oplus R^n \simeq R^{n+1}$  for some  $n$ . We will now need to construct the **exterior powers**, for which we digress:

**Definition 4.7** Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Then  $\wedge M$ , the **exterior algebra on  $M$** , is the free (noncommutative) graded  $R$ -algebra generated by  $M$  (with product  $\wedge$ ) with just enough relations such that  $\wedge$  is anticommutative (and, *more strongly*,  $x \wedge x = 0$  for  $x$  degree one).

Clearly  $\wedge M$  is a quotient of the **tensor algebra**  $T(M)$ , which is by definition  $R \oplus M \oplus M \otimes M \oplus \cdots \oplus M^{\otimes n} \oplus \cdots$ . The tensor algebra is a graded  $R$ -algebra in an obvious way:  $(x_1 \otimes \cdots \otimes x_a) \cdot (y_1 \otimes \cdots \otimes y_b) = x_1 \otimes \cdots \otimes x_a \otimes y_1 \otimes \cdots \otimes y_b$ . This is an associative  $R$ -algebra. Then

$$\wedge M = T(M)/(x \otimes x, x, y \in M).$$

The grading on  $\wedge M$  comes from the grading of  $T(M)$ .

We are interested in basically one example:

**Example 4.8** Say  $M = R^m$ . Then  $\wedge^m M = R$ . If  $e_1, \dots, e_m \in M$  are generators, then  $e_1 \wedge \dots \wedge e_m$  is a generator. More generally,  $\wedge^k M$  is free on  $e_{i_1} \wedge \dots \wedge e_{i_k}$  for  $i_1 < \dots < i_k$ .

We now make:

**Definition 4.9** If  $M$  is a projective  $R$ -module of rank  $n$ , then

$$\det(M) = \wedge^n M.$$

If  $M$  is free, then  $\det(M)$  is free of rank one. So, as we see by localization,  $\det(M)$  is always an invertible module for  $M$  locally free (i.e. projective) and  $\wedge^{n+1} M = 0$ .

**Lemma 4.10**  $\det(M \oplus N) = \det M \otimes \det N$ .

*Proof.* This isomorphism is given by wedging  $\wedge^{\text{top}} M \otimes \wedge^{\text{top}} N \rightarrow \wedge^{\text{top}}(M \oplus N)$ . This is easily checked for oneself. ▲

Anyway, let us finally go back to the proof. If  $I \oplus R^n = R^{n+1}$ , then taking determinants shows that

$$\det I \otimes R = R, \quad \text{▲}$$

so  $\det I = R$ . But this is  $I$  as  $I$  is of rank one. So  $I$  is free.

**Theorem 4.11** *A regular local ring is factorial.*

Let  $R$  be a regular local ring of dimension  $n$ . We want to show that  $R$  is factorial. Choose a prime ideal  $\mathfrak{p}$  of height one. We'd like to show that  $\mathfrak{p}$  is principal.

*Proof.* Induction on  $n$ . If  $n = 0$ , then we are done—we have a field.

If  $n = 1$ , then a height one prime is maximal, hence principal, because regularity is equivalent to the ring's being a DVR.

Assume  $n > 1$ . The prime ideal  $\mathfrak{p}$  has height one, so it is contained in a maximal ideal  $\mathfrak{m}$ . Note that  $\mathfrak{m}^2 \subset \mathfrak{m}$  as well. I claim that there is an element  $x$  of  $\mathfrak{m} - \mathfrak{p} - \mathfrak{m}^2$ . This follows as an argument like prime avoidance. To see that  $x$  exists, choose  $x_1 \in \mathfrak{m} - \mathfrak{p}$  and  $x_2 \in \mathfrak{m} - \mathfrak{m}^2$ . We are done unless  $x_1 \in \mathfrak{m}^2$  and  $x_2 \in \mathfrak{p}$  (or we could take  $x$  to be  $x_1$  or  $x_2$ ). In this case, we just take  $x = x_1 + x_2$ .

So choose  $x \in \mathfrak{m} - \mathfrak{p} - \mathfrak{m}^2$ . Let us examine the ring  $R_x = R[1/x]$ , which contains an ideal  $\mathfrak{p}[x^{-1}]$ . This is a proper ideal as  $x \notin \mathfrak{p}$ . Now  $R[1/x]$  is regular (i.e. its localizations at primes are regular local). The dimension, however, is of dimension less than  $n$  since by inverting  $x$  we have removed  $\mathfrak{m}$ . By induction we can assume that  $R_x$  is locally factorial.

Now  $\mathfrak{p}R_x$  is prime and of height one, so it is invertible as  $R_x$  is locally factorial. In particular it is projective.

But  $\mathfrak{p}$  has a finite resolution by  $R$ -modules (by regularity), so  $\mathfrak{p}R_x$  has a finite free resolution. In particular,  $\mathfrak{p}R_x$  is stably free and invertible, hence free. Thus  $\mathfrak{p}R_x$  is **principal**.

We want to show that  $\mathfrak{p}$  is principal, not just after localization. We know that there is a  $y \in \mathfrak{p}$  such that  $y$  generates  $\mathfrak{p}R_x$ . Choose  $y$  such that  $(y) \subset \mathfrak{p}$  is as large as possible. We can do this since  $R$  is noetherian. This implies that  $x \nmid y$  because otherwise we could use  $y/x$  instead of  $y$ .

We shall now show that

$$\mathfrak{p} = (y).$$

So suppose  $z \in \mathfrak{p}$ . We know that  $y$  generates  $\mathfrak{p}$  **after  $x$  is inverted**. In particular,  $z \in \mathfrak{p}R_x$ . That is,  $zx^a \in (y)$  for a large. That is, we can write

$$zx^a = yw, \quad \text{for some } w \in R.$$

We chose  $x$  such that  $x \notin \mathfrak{m}^2$ . In particular,  $R/(x)$  is regular, hence an integral domain; i.e.  $x$  is a prime element. We find that  $x$  must divide one of  $y, w$  if  $a > 0$ . But we know that  $x \nmid y$ , so  $x \mid w$ . Thus  $w = w'x$  for some  $x$ . We find that, cancelling  $x$ ,

$$zx^{a-1} = yw'$$

and we can repeat this argument over and over until we find that

$$z \in (y).$$

▲

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