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Chapter 6
Graded and filtered rings

In algebraic geometry, working in classical affine space $\mathbb{A}^n_{\mathbb{C}}$ of points in $\mathbb{C}^n$ turns out to be insufficient for various reasons. Instead, it is often more convenient to consider varieties in *projective space* $\mathbb{P}^n_{\mathbb{C}}$, which is the set of lines through the origin in $\mathbb{C}^{n+1}$. In other words, it is the set of all $n+1$-tuples $[z_0, \dots, z_n] \in \mathbb{C}^{n+1} - \{0\}$ modulo the relation that

$$[z_0, \dots, z_n] = [\lambda z_0, \dots, \lambda z_n], \quad \lambda \in \mathbb{C}^*. \tag{6.1}$$

Varieties in projective space have many convenient properties that affine varieties do not: for instance, intersections work out much more nicely when intersections at the extra "points at infinity" are included. Moreover, when endowed with the complex topology, (complex) projective varieties are *compact*, unlike all but degenerate affine varieties (i.e. finite sets).

It is when defining the notion of a "variety" in projective space that one encounters gradedness. Now a variety in $\mathbb{P}^n$ must be cut out by polynomials $F_1, \dots, F_k \in \mathbb{C}[x_0, \dots, x_n]$; that is, a point represented by $[z_0, \dots, z_n]$ lies in the associated variety if and only if $F_i(z_0, \dots, z_n) = 0$ for each i. For this to make sense, or to be independent of the choice of $z_0, \dots, z_n$ up to rescaling as in (6.1), it is necessary to assume that each F_i is *homogeneous*.

Algebraically, $\mathbb{A}^n_{\mathbb{C}}$ is the set of maximal ideals in the polynomial ring $\mathbb{C}^n$. Projective space is defined somewhat more geometrically (as a set of lines) but it turns out that there is an algebraic interpretation here too. The points of projective space are in bijection with the *homogeneous maximal ideals* of the polynomial ring $\mathbb{C}[x_0, \dots, x_n]$. We shall define more generally the Proj of a *graded* ring in this chapter. Although we shall not repeatedly refer to this concept in the sequel, it will be useful for readers interested in algebraic geometry.

We shall also introduce the notion of a *filtration*. A filtration allows one to endow a given module with a topology, and one can in fact complete with respect to this topology. This construction will be studied in Chapter 11.

§1 Graded rings and modules

Much of the material in the present section is motivated by algebraic geometry; see [GD], volume II for the construction of $\operatorname{Proj} R$ as a scheme.

1.1 Basic definitions

Definition 1.1 A **graded ring** R is a ring together with a decomposition (as abelian groups)

$$R = R_0 \oplus R_1 \oplus \dots$$

such that $R_m R_n \subset R_{m+n}$ for all $m, n \in \mathbb{Z}_{\geq 0}$, and where R_0 is a subring (i.e. $1 \in R_0$). A $\mathbb{Z}$**-graded ring** is one where the decomposition is into $\bigoplus_{n \in \mathbb{Z}} R_n$. In either case, the elements of the subgroup R_n are called **homogeneous of degree** n.

The basic example to keep in mind is, of course, the polynomial ring $R[x_1, \dots, x_n]$ for R any ring. The graded piece of degree n consists of the homogeneous polynomials of degree n.

Consider a graded ring R.

Definition 1.2 A **graded** R-module is an ordinary R-module M together with a decomposition

$$M = \bigoplus_{k \in \mathbb{Z}} M_k$$

as abelian groups, such that $R_m M_n \subset M_{m+n}$ for all $m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}$. Elements in one of these pieces are called **homogeneous.** Any $m \in M$ is thus uniquely a finite sum $\sum m_{n_i}$ where each $m_{n_i} \in M_{n_i}$ is homogeneous of degree n_i.

Clearly there is a *category* of graded R-modules, where the morphisms are the morphisms of R-modules that preserve the grading (i.e. take homogeneous elements to homogeneous elements of the same degree).

Since we shall focus on positively graded rings, we shall simply call them graded rings; when we do have to consider rings with possibly negative gradings, we shall highlight this explicitly. Note, however, that we allow modules with negative gradings freely.

In fact, we shall note an important construction that will generally shift the graded pieces such that some of them might be negative:

Definition 1.3 Given a graded module M, we define the **twist** $M(n)$ as the same R-module but with the grading

$$M(n)_k = M_{n+k}.$$

This is a functor on the category of graded R-modules.

In algebraic geometry, the process of twisting allows one to construct canonical line bundles on projective space. Namely, a twist of R itself will lead to a line bundle on projective space that in general is not trivial. See [Har77], II.5.

Here are examples:

Example 1.4 (An easy example) If R is a graded ring, then R is a graded module over itself.

Example 1.5 (Another easy example) If S is any ring, then S can be considered as a graded ring with $S_0 = S$ and $S_i = 0$ for $i > 0$. Then a *graded* S-module is just a $\mathbb{Z}$-indexed collection of (ordinary) S-modules.

Example 1.6 (The blowup algebra) This example is a bit more interesting, and will be used in the sequel. Let S be any ring, and let $J \subset S$ be an ideal. We can make $R = S \oplus J \oplus J^2 \oplus \dots$ (the so-called *blowup algebra*) into a graded ring, by defining the multiplication the normal way except that something in the ith component times something in the jth component goes into the $i+j$th component.

Given any S-module M, there is a graded R-module $M \oplus JM \oplus J^2 M \oplus \dots$, where multiplication is defined in the obvious way. We thus get a functor from S-modules to graded R-modules.

Definition 1.7 Fix a graded ring R. Let M be a graded R-module and $N \subset M$ an R-submodule. Then N is called a **graded submodule** if the homogeneous components of anything in N are in N. If $M = R$, then a graded ideal is also called a **homogeneous ideal**.

In particular, a graded submodule is automatically a graded module in its own right.

Lemma 1.8 1. *The sum of two graded submodules (in particular, homogeneous ideals) is graded.*

2. The intersection of two graded submodules is graded.

Proof. Immediate. ▲

One can grade the quotients of a graded module by a graded submodule. If $N \subset M$ is a graded submodule, then M/N can be made into a graded module, via the isomorphism of abelian groups

$$M/N \simeq \bigoplus_{k \in \mathbb{Z}} M_k/N_k.$$

In particular, if $\mathfrak{a} \subset R$ is a homogeneous ideal, then $R/\mathfrak{a}$ is a graded ring in a natural way.

EXERCISE 6.1 Let R be a graded ring. Does the category of graded R-modules admit limits and colimits?

1.2 Homogeneous ideals

Recall that a homogeneous ideal in a graded ring R is simply a graded submodule of R. We now prove a useful result that enables us tell when an ideal is homogeneous.

Proposition 1.9 *Let R be a graded ring, $I \subset R$ an ideal. Then I is a homogeneous ideal if and only if it can be generated by homogeneous elements.*

Proof. If I is a homogeneous ideal, then by definition

$$I = \bigoplus_i I \cap R_i,$$

so I is generated by the sets $\{I \cap R_i\}_{i \in \mathbb{Z}_{\geq 0}}$ of homogeneous elements.

Conversely, let us suppose that I is generated by homogeneous elements $\{h_\alpha\}$. Let $x \in I$ be arbitrary; we can uniquely decompose x as a sum of homogeneous elements, $x = \sum x_i$, where each $x_i \in R_i$. We need to show that each $x_i \in I$ in fact.

To do this, note that $x = \sum q_\alpha h_\alpha$ where the q_α belong to R. If we take ith homogeneous components, we find that

$$x_i = \sum (q_\alpha)_{i - \deg h_\alpha} h_\alpha,$$

where $(q_\alpha)_{i-\deg h_\alpha}$ refers to the homogeneous component of q_α concentrated in the degree $i - \deg h_\alpha$. From this it is easy to see that each x_i is a linear combination of the h_α and consequently lies in I. ▲

Example 1.10 If $\mathfrak{a}, \mathfrak{b} \subset R$ are homogeneous ideals, then so is $\mathfrak{ab}$. This is clear from Proposition 1.9.

Example 1.11 Let k be a field. The ideal $(x^2 + y)$ in $k[x, y]$ is *not* homogeneous. However, we find from Proposition 1.9 that the ideal $(x^2 + y^2, y^3)$ is.

Since we shall need to use them to define $\operatorname{Proj} R$ in the future, we now prove a result about homogeneous *prime* ideals specifically. Namely, "primeness" can be checked just on homogeneous elements for a homogeneous ideal.

Lemma 1.12 *Let $\mathfrak{p} \subset R$ be a homogeneous ideal. In order that $\mathfrak{p}$ be prime, it is necessary and sufficient that whenever x, y are* homogeneous *elements such that $xy \in \mathfrak{p}$, then at least one of $x, y \in \mathfrak{p}$.*

Proof. Necessity is immediate. For sufficiency, suppose $a, b \in R$ and $ab \in \mathfrak{p}$. We must prove that one of these is in $\mathfrak{p}$. Write

$$a = a_{k_1} + a_1 + \dots + a_{k_2}, \ b = b_{m_1} + \dots + b_{m_2}$$

as a decomposition into homogeneous components (i.e. a_i is the ith component of a), where a_{k_2}, b_{m_2} are nonzero and of the highest degree.

Let $k = k_2 - k_1, m = m_2 - m_1$. So there are k homogeneous terms in the expression for a, m in the expression for b. We will prove that one of $a, b \in \mathfrak{p}$ by induction on $m + n$. When $m + n = 0$, then it is just the condition of the lemma. Suppose it true for smaller values of $m + n$. Then ab has highest homogeneous component $a_{k_2} b_{m_2}$, which must be in $\mathfrak{p}$ by homogeneity. Thus one of a_{k_2}, b_{m_2} belongs to $\mathfrak{p}$. Say for definiteness it is a_k. Then we have that

$$(a - a_{k_2})b \equiv ab \equiv 0 \text{ mod } \mathfrak{p}$$

so that $(a - a_{k_2})b \in \mathfrak{p}$. But the resolutions of $a - a_{k_2}, b$ have a smaller $m + n$-value: $a - a_{k_2}$ can be expressed with $k - 1$ terms. By the inductive hypothesis, it follows that one of these is in $\mathfrak{p}$, and since $a_k \in \mathfrak{p}$, we find that one of $a, b \in \mathfrak{p}$. ▲

1.3 Finiteness conditions

There are various finiteness conditions (e.g. noetherianness) that one often wants to impose in algebraic geometry. Since projective varieties (and schemes) are obtained from graded rings, we briefly discuss these finiteness conditions for them.

Definition 1.13 For a graded ring R, write $R_+ = R_1 \oplus R_2 \oplus \dots$. Clearly $R_+ \subset R$ is a homogeneous ideal. It is called the **irrelevant ideal.**

When we define the Proj of a ring, prime ideals containing the irrelevant ideal will be no good. The intuition is that when one is working with $\mathbb{P}^n_{\mathbb{C}}$, the irrelevant ideal in the corresponding ring $\mathbb{C}[x_0, \dots, x_n]$ corresponds to *all* homogeneous polynomials of positive degree. Clearly these have no zeros except for the origin, which is not included in projective space: thus the common zero locus of the irrelevant ideal should be $\emptyset \subset \mathbb{P}^n_{\mathbb{C}}$.

Proposition 1.14 *Suppose $R = R_0 \oplus R_1 \oplus \dots$ is a graded ring. Then if a subset $S \subset R_+$ generates the irrelevant ideal R_+ as R-ideal, it generates R as R_0-algebra.*

The converse is clear as well. Indeed, if $S \subset R_+$ generates R as an R_0-algebra, clearly it generates R_+ as an R-ideal.

Proof. Let $T \subset R$ be the R_0-algebra generated by S. We shall show inductively that $R_n \subset T$. This is true for $n = 0$. Suppose $n > 0$ and the assertion true for smaller n. Then, we have

$$\begin{aligned} R_n &= RS \cap R_n \text{ by assumption} \\ &= (R_0 \oplus R_1 \oplus \dots \oplus R_{n-1})(S) \cap R_n \text{ because } S \subset R_+ \\ &\subset (R_0[S])(S) \cap R_n \text{ by inductive hypothesis} \\ &\subset R_0(S). \end{aligned}$$

▲

Theorem 1.15 *The graded ring R is noetherian if and only if R_0 is noetherian and R is finitely generated as R_0-algebra.*

Proof. One direction is clear by Hilbert's basis theorem. For the other, suppose R noetherian. Then R_0 is noetherian because any sequence $I_1 \subset I_2 \subset \dots$ of ideals of R_0 leads to a sequence of ideals $I_1 R \subset I_2 R \subset \dots$, and since these stabilize, the original $I_1 \subset I_2 \subset \dots$ must stabilize too. (Alternatively, $R_0 = R/R_+$, and taking quotients preserves noetherianness.) Moreover, since R_+ is a finitely generated R-ideal by noetherianness, it follows that R is a finitely generated R_0-algebra too: we can, by Proposition 1.14, take as R_0-algebra generators for R a set of generators for the *ideal* R_+. ▲

The basic finiteness condition one often needs is that R should be finitely generated as an R_0-algebra. We may also want to have that R is generated by R_1, quite frequently—in algebraic geometry, this implies a bunch of useful things about certain sheaves being invertible. (See [GD], volume II.2.) As one example, having R generated as R_0-algebra by R_1 is equivalent to having R a *graded* quotient of a polynomial algebra over R_0 (with the usual grading). Geometrically, this equates to having $\operatorname{Proj} R$ contained as a closed subset of some projective space over R_0.

However, sometimes we have the first condition and not the second, though if we massage things we can often assure generation by R_1. Then the next idea comes in handy.

Definition 1.16 Let R be a graded ring and $d \in \mathbb{N}$. We set $R^{(d)} = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} R_{kd}$; this is a graded ring and R_0-algebra. If M is a graded R-module and $l \in \{0, 1, \dots, d-1\}$, we write $M^{(d,l)} = \bigoplus_{k \equiv l \bmod d} M_k$. Then $M^{(d,l)}$ is a graded $R^{(d)}$-module.

We in fact have a functor $\cdot^{(d,l)}$ from graded R-modules to graded $R^{(d)}$-modules.

One of the implications of the next few results is that, by replacing R with $R^{(d)}$, we can make the condition "generated by terms of degree 1" happen. But first, we show that basic finiteness is preserved if we filter out some of the terms.

Proposition 1.17 *Let R be a graded ring and a finitely generated R_0-algebra. Let M be a finitely generated R-module.*

1. *Each M_i is finitely generated over R_0, and the M_i become zero when $i \ll 0$.*
2. *$M^{(d,l)}$ is a finitely generated $R^{(d)}$ module for each d, l. In particular, M itself is a finitely generated $R^{(d)}$-module.*
3. *$R^{(d)}$ is a finitely generated R_0-algebra.*

Proof. Choose homogeneous generators $m_1, \dots, m_k \in M$. For instance, we can choose the homogeneous components of a finite set of generators for M. Then every nonzero element of M has degree at least $\min(\deg m_i)$. This proves the last part of (1). Moreover, let $r_1, \dots, r_p$ be algebra generators of R over R_0. We can assume that these are homogeneous with positive degrees $d_1, \dots, d_p > 0$. Then the R_0-module M_i is generated by the elements

$$r_1^{a_1} \dots r_p^{a_p} m_s$$

where $\sum a_j d_j + \deg m_s = i$. Since the $d_j > 0$ and there are only finitely many m_s's, there are only finitely many such elements. This proves the rest of (1).

To prove (2), note first that it is sufficient to show that M is finitely generated over $R^{(d)}$, because the $M^{(d,l)}$ are $R^{(d)}$-homomorphic images (i.e. quotient by the $M^{(d',l)}$ for $d' \neq d$). Now M is generated as R_0-module by the $r_1^{a_1} \dots r_p^{a_p} m_s$ for $a_1, \dots, a_p \geq 0$ and $s = 1, \dots, k$. In particular, by the euclidean algorithm in elementary number theory, it follows that the $r_1^{a_1} \dots r_p^{a_p} m_s$ for $a_1, \dots, a_p \in [0, d-1]$ and $s = 1, \dots, k$ generate M over $R^{(d)}$, as each power $r_i^d \in R^{(d)}$. In particular, R is finitely generated over $R^{(d)}$.

When we apply (2) to the finitely generated R-module R_+, it follows that $R_+^{(d)}$ is a finitely generated $R^{(d)}$-module. This implies that $R^{(d)}$ is a finitely generated R_0-algebra by Proposition 1.14. ▲

In particular, by **??** (later in the book!) R is *integral* over $R^{(d)}$: this means that each element of R satisfies a monic polynomial equation with $R^{(d)}$-coefficients. This can easily be seen directly. The dth power of a homogeneous element lies in $R^{(d)}$.

Remark Part (3), the preservation of the basic finiteness condition, could also be proved as follows, at least in the noetherian case (with $S = R^{(d)}$). We shall assume familiarity with the material in **??** for this brief digression.

Lemma 1.18 *Suppose $R_0 \subset S \subset R$ is an inclusion of rings with R_0 noetherian. Suppose R is a finitely generated R_0-algebra and R/S is an integral extension. Then S is a finitely generated R_0-algebra.*

In the case of interest, we can take $S = R^{(d)}$. The point of the lemma is that finite generation can be deduced for *subrings* under nice conditions.

Proof. We shall start by finding a subalgebra $S' \subset S$ such that R is integral over S', but S' is a finitely generated R_0-algebra. The procedure will be a general observation of the flavor of "noetherian descent" to be developed in **??**. Then, since R is integral over S' and finitely generated as an *algebra*, it will be finitely generated as a S'-module. S, which is a sub-S'-module, will equally be finitely generated as a S'-module, hence as an R_0-algebra. So the point is to make S finitely generated as a module over a "good" ring.

Indeed, let $r_1, \dots, r_m$ be generators of R/R_0. Each satisfies an integral equation $r_k^{n_k} + P_k(r_k) = 0$, where $P_k \in S[X]$ has degree less than n_k. Let $S' \subset S \subset R$ be the subring generated over R_0 by the coefficients of all these polynomials P_k.

Then R is, by definition, integral over S'. Since R is a finitely generated S'-algebra, it follows by **??** that it is a finitely generated S'-module. Then S, as a S'-submodule is a finitely generated S'-module by noetherianness. Therefore, S is a finitely generated R_0-algebra. ▲

This result implies, incidentally, the following useful corollary:

Corollary 1.19 *Let R be a noetherian ring. If a finite group G acts on a finitely generated R-algebra S, the ring of invariants S^G is finitely generated.*

Proof. Apply Lemma 1.18 to R, S^G, S. One needs to check that S is integral over S^G. But each $s \in S$ satisfies the equation

$$\prod_{\sigma \in G} (X - \sigma(s)),$$

which has coefficients in S^G. ▲

This ends the digression.

We next return to our main goals, and let R be a graded ring, finitely generated as an R_0-algebra, as before; let M be a finitely generated R-module. We show that we can have $R^{(d)}$ generated by terms of degree d (i.e. "degree 1" if we rescale) for d chosen large.

Lemma 1.20 *Hypotheses as above, there is a pair (d, n_0) such that*

$$R_d M_n = M_{n+d}$$

for $n \geq n_0$.

Proof. Indeed, select R-module generators $m_1, \dots, m_k \in M$ and R_0-algebra generators $r_1, \dots, r_p \in R$ as in the proof of Proposition 1.17; use the same notation for their degrees, i.e. $d_j = \deg r_j$. Let d be the least common multiple of the d_j. Consider the family of elements

$$s_i = r_i^{d/d_i} \in R_d.$$

Then suppose $m \in M_n$ for $n > d + \sup \deg m_i$. We have that m is a sum of products of powers of the $\{r_j\}$ and the $\{m_i\}$, each term of which we can assume is of degree n. In this case, since in each term, at least one of the $\{r_j\}$ must occur to power $\geq \frac{d}{d_j}$, we can write each term in the sum as some s_j times something in M_{n-d}.

In particular, $M_n = R_d M_{n-d}$. ▲

Proposition 1.21 *Suppose R is a graded ring and finitely generated R_0-algebra. Then there is $d \in \mathbb{N}$ such that $R^{(d)}$ is generated over R_0 by R_d.*

What this proposition states geometrically is that if we apply the functor $R \mapsto R^{(d)}$ for large d (which, geometrically, is actually harmless), one can arrange things so that $\operatorname{Proj} R$ (not defined yet!) is contained as a closed subscheme of ordinary projective space.

Proof. Consider R as a finitely generated, graded R-module. Suppose d' is as in the Proposition 1.21 (replacing d, which we reserve for something else), and choose n_0 accordingly. So we have $R_{d'} R_m = R_{m+d'}$ whenever $m \geq n_0$. Let d be a multiple of d' which is greater than n_0.

Then, iterating, we have $R_d R_n = R_{d+n}$ if $n \geq d$ since d is a multiple of d'. In particular, it follows that $R_{nd} = (R_d)^n$ for each $n \in \mathbb{N}$, which implies the statement of the proposition. ▲

As we will see below, taking $R^{(d)}$ does not affect the Proj, so this is extremely useful.

Example 1.22 Let k be a field. Then $R = k[x^2] \subset k[x]$ (with the grading induced from $k[x]$) is a finitely generated graded k-algebra, which is not generated by its elements in degree one (there are none!). However, $R^{(2)} = k[x^2]$ is generated by x^2.

We next show that taking the $R^{(d)}$ *always* preserves noetherianness.

Proposition 1.23 *If R is noetherian, then so is $R^{(d)}$ for any $d > 0$.*

Proof. If R is noetherian, then R_0 is noetherian and R is a finitely generated R_0-algebra by Theorem 1.15. Proposition 1.17 now implies that $R^{(d)}$ is also a finitely generated R_0-algebra, so it is noetherian. ▲

The converse is also true, since R is a finitely generated $R^{(d)}$-module.

1.4 Localization of graded rings

Next, we include a few topics that we shall invoke later on. First, we discuss the interaction of homogeneity and localization. Under favorable circumstances, we can give $\mathbb{Z}$-gradings to localizations of graded rings.

Definition 1.24 If $S \subset R$ is a multiplicative subset of a graded (or $\mathbb{Z}$-graded) ring R consisting of homogeneous elements, then $S^{-1}R$ is a $\mathbb{Z}$-graded ring: we let the homogeneous elements of degree n be of the form r/s where $r \in R_{n+\deg s}$. We write $R_{(S)}$ for the subring of elements of degree zero; there is thus a map $R_0 \to R_{(S)}$.

If S consists of the powers of a homogeneous element f, we write $R_{(f)}$ for R_S. If $\mathfrak{p}$ is a homogeneous ideal and S the set of homogeneous elements of R not in $\mathfrak{p}$, we write $R_{(\mathfrak{p})}$ for $R_{(S)}$.

Of course, $R_{(S)}$ has a trivial grading, and is best thought of as a plain, unadorned ring. We shall show that $R_{(f)}$ is a special case of something familiar.

Proposition 1.25 *Suppose f is of degree d. Then, as plain rings, there is a canonical isomorphism $R_{(f)} \simeq R^{(d)}/(f-1)$.*

Proof. The homomorphism $R^{(d)} \to R_{(f)}$ is defined to map $g \in R_{kd}$ to $g/f^d \in R_{(f)}$. This is then extended by additivity to non-homogeneous elements. It is clear that this is multiplicative, and that the ideal $(f-1)$ is annihilated by the homomorphism. Moreover, this is surjective.

We shall now define an inverse map. Let $x/f^n \in R_{(f)}$; then x must be a homogeneous element of degree divisible by d. We map this to the residue class of x in $R^{(d)}/(f-1)$. This is well-defined; if $x/f^n = y/f^m$, then there is N with

$$f^N(xf^m - yf^n) = 0,$$

so upon reduction (note that f gets reduced to 1!), we find that the residue classes of x, y are the same, so the images are the same.

Clearly this defines an inverse to our map. ▲

Corollary 1.26 *Suppose R is a graded noetherian ring. Then each of the $R_{(f)}$ is noetherian.*

Proof. This follows from the previous result and the fact that $R^{(d)}$ is noetherian (Proposition 1.23).

More generally, we can define the localization procedure for graded modules.

Definition 1.27 Let M be a graded R-module and $S \subset R$ a multiplicative subset consisting of homogeneous elements. Then we define $M_{(S)}$ as the submodule of the graded module $S^{-1}M$ consisting of elements of degree zero. When S consists of the powers of a homogeneous element $f \in R$, we write $M_{(f)}$ instead of $M_{(S)}$. We similarly define $M_{(\mathfrak{p})}$ for a homogeneous prime ideal $\mathfrak{p}$.

Then clearly $M_{(S)}$ is a $R_{(S)}$-module. This is evidently a functor from graded R-modules to $R_{(S)}$-modules.

We next observe that there is a generalization of Proposition 1.25.

Proposition 1.28 *Suppose M is a graded R-module, $f \in R$ homogeneous of degree d. Then there is an isomorphism*

$$M_{(f)} \simeq M^{(d)}/(f-1)M^{(d)}$$

of $R^{(d)}$-modules.

Proof. This is proved in the same way as Proposition 1.25. Alternatively, both are right-exact functors that commute with arbitrary direct sums and coincide on R, so must be naturally isomorphic by a well-known bit of abstract nonsense.[1] ▲

In particular:

Corollary 1.29 *Suppose M is a graded R-module, $f \in R$ homogeneous of degree 1. Then we have*

$$M_{(f)} \simeq M/(f-1)M \simeq M \otimes_R R/(f-1).$$

1.5 The Proj of a ring

Let $R = R_0 \oplus R_1 \oplus \dots$ be a **graded ring**.

Definition 1.30 Let $\operatorname{Proj} R$ denote the set of *homogeneous prime ideals* of R that do not contain the **irrelevant ideal** R_+.[2]

[1] Citation needed.

[2] Recall that an ideal $\mathfrak{a} \subset R$ for R graded is *homogeneous* if the homogeneous components of $\mathfrak{a}$ belong to $\mathfrak{a}$.

We can put a topology on $\operatorname{Proj} R$ by setting, for a homogeneous ideal $\mathfrak{b}$,

$$V(\mathfrak{b}) = \{\mathfrak{p} \in \operatorname{Proj} R : \mathfrak{p} \supset \mathfrak{b}\}$$

. These sets satisfy

1. $V(\sum \mathfrak{b}_\mathfrak{i}) = \bigcap V(\mathfrak{b}_\mathfrak{i})$.
2. $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.
3. $V(\operatorname{Rad} \mathfrak{a}) = V(\mathfrak{a})$.

Note incidentally that we would not get any more closed sets if we allowed all ideals $\mathfrak{b}$, since to any $\mathfrak{b}$ we can consider its "homogenization." We could even allow all sets.

In particular, the V's do in fact yield a topology on $\operatorname{Proj} R$ (setting the open sets to be complements of the V's). As with the affine case, we can define basic open sets. For f homogeneous of positive degree, define $D'(f)$ to be the collection of homogeneous ideals (not containing R_+) that do not contain f; clearly these are open sets.

Let $\mathfrak{a}$ be a homogeneous ideal. Then we claim that:

Lemma 1.31 $V(\mathfrak{a}) = V(\mathfrak{a} \cap R_+)$.

Proof. Indeed, suppose $\mathfrak{p}$ is a homogeneous prime not containing S_+ such that all homogeneous elements of positive degree in $\mathfrak{a}$ (i.e., anything in $\mathfrak{a} \cap R_+$) belongs to $\mathfrak{p}$. We will show that $\mathfrak{a} \subset \mathfrak{p}$.

Choose $a \in \mathfrak{a} \cap R_0$. It is sufficient to show that any such a belongs to $\mathfrak{p}$ since we are working with homogeneous ideals. Let f be a homogeneous element of positive degree that is not in $\mathfrak{p}$. Then $af \in \mathfrak{a} \cap R_+$, so $af \in \mathfrak{p}$. But $f \notin \mathfrak{p}$, so $a \in \mathfrak{p}$. ▲

Thus, when constructing these closed sets $V(\mathfrak{a})$, it suffices to work with ideals contained in the irrelevant ideal. In fact, we could take $\mathfrak{a}$ in any prescribed power of the irrelevant ideal, since taking radicals does not affect V.

Proposition 1.32 *We have $D'(f) \cap D'(g) = D'(fg)$. Also, the $D'(f)$ form a basis for the topology on* $\operatorname{Proj} R$.

Proof. The first part is evident, by the definition of a prime ideal. We prove the second. Note that $V(\mathfrak{a})$ is the intersection of the $V((f))$ for the homogeneous $f \in \mathfrak{a} \cap R_+$. Thus $\operatorname{Proj} R - V(\mathfrak{a})$ is the union of these $D'(f)$. So every open set is a union of sets of the form $D'(f)$. ▲

We shall now show that the topology is actually rather familiar from the affine case, which is not surprising, since the definition is similar.

Proposition 1.33 *$D'(f)$ is homeomorphic to* $\operatorname{Spec} R_{(f)}$ *under the map*

$$\mathfrak{p} \to \mathfrak{p}R_f \cap R_{(f)}$$

sending homogeneous prime ideals of R not containing f into primes of $R_{(f)}$.

Proof. Indeed, let $\mathfrak{p}$ be a homogeneous prime ideal of R not containing f. Consider $\phi(\mathfrak{p}) = \mathfrak{p}R_f \cap R_{(f)}$ as above. This is a prime ideal, since $\mathfrak{p}R_f$ is a prime ideal in R_f by basic properties of localization, and $R_{(f)} \subset R_f$ is a subring. (It cannot contain the identity, because that would imply that a power of f lay in $\mathfrak{p}$.)

So we have defined a map $\phi : D'(f) \to \operatorname{Spec} R_{(f)}$. We can define its inverse ψ as follows. Given $\mathfrak{q} \subset R_{(f)}$ prime, we define a prime ideal $\mathfrak{p} = \psi(\mathfrak{q})$ of R by saying that a *homogeneous* element $x \in R$ belongs to $\mathfrak{p}$ if and only if $x^{\deg f}/f^{\deg x} \in \mathfrak{q}$. It is easy to see that this is indeed an ideal, and that it is prime by Lemma 1.12.

Furthermore, it is clear that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity. This is because $x \in \mathfrak{p}$ for $\mathfrak{p} \in D'(f)$ if and only if $f^n x \in \mathfrak{p}$ for some n.

We next need to check that these are continuous, hence homeomorphisms. If $\mathfrak{a} \subset R$ is a homogeneous ideal, then $V(\mathfrak{a}) \cap D'(f)$ is mapped to $V(\mathfrak{a}R_f \cap R_{(f)}) \subset \operatorname{Spec} R_{(f)}$, and vice versa. ▲

§2 Filtered rings

In practice, one often has something weaker than a grading. Instead of a way of saying that an element is of degree d, one simply has a way of saying that an element is "of degree at most d." This leads to the definition of a *filtered* ring (and a filtered module). We shall use this definition in placing topologies on rings and modules and, later, completing them.

2.1 Definition

Definition 2.1 A **filtration** on a ring R is a sequence of ideals $R = I_0 \supset I_1 \supset \dots$ such that $I_m I_n \subset I_{m+n}$ for each $m, n \in \mathbb{Z}_{\geq 0}$. A ring with a filtration is called a **filtered ring**.

A filtered ring is supposed to be a generalization of a graded ring. If $R = \bigoplus R_k$ is graded, then we can make R into a filtered ring in a canonical way by taking the ideal $I_m = \bigoplus_{k \geq m} R_k$ (notice that we are using the fact that R has only pieces in nonnegative gradings!).

We can make filtered rings into a category: a morphism of filtered rings $\phi : R \to S$ is a ring-homomorphism preserving the filtration.

Example 2.2 (The *I*-adic filtration) Given an ideal $I \subset R$, we can take powers of I to generate a filtration. This filtration $R \supset I \supset I^2 \supset \dots$ is called the ***I*-adic filtration,** and is especially important when R is local and I the maximal ideal.

If one chooses the polynomial ring $k[x_1, \dots, x_n]$ over a field with n variables and takes the $(x_1, \dots, x_n)$-adic filtration, one gets the same as the filtration induced by the usual grading.

Example 2.3 As a specialization of the previous example, consider the power series ring $R = k[[x]]$ over a field k with one indeterminate x. This is a local ring (with maximal ideal (x)), and it has a filtration with $R_i = (x^i)$. Note that this ring, unlike the polynomial ring, is *not* a graded ring in any obvious way.

When we defined graded rings, the first thing we did thereafter was to define the notion of a graded module over a graded ring. We do the analogous thing for filtered modules.

Definition 2.4 Let R be a filtered ring with a filtration $I_0 \supset I_1 \supset \dots$. A **filtration** on an R-module M is a decreasing sequence of submodules

$$M = M_0 \supset M_1 \supset M_2 \supset \dots$$

such that $I_m M_n \subset M_{n+m}$ for each m, n. A module together with a filtration is called a **filtered module.**

As usual, there is a category of filtered modules over a fixed filtered ring R, with morphisms the module-homomorphisms that preserve the filtrations.

Example 2.5 (The *I*-adic filtration for modules) Let R be any ring and $I \subset R$ any ideal. Then if we make R into a filtered ring with the I-adic filtration, we can make any R-module M into a filtered R-module by giving M the filtration

$$M \supset IM \supset I^2 M \supset \dots,$$

which is also called the ***I*-adic filtration.**

2.2 The associated graded

We shall now describe a construction that produces graded things from filtered ones.

Definition 2.6 Given a filtered ring R (with filtration $\{I_n\}$), the **associated graded ring** $\mathrm{gr}(R)$ is the graded ring

$$\mathrm{gr}(R) = \bigoplus_{n=0}^{\infty} I_n/I_{n+1}.$$

This is made into a ring by the following procedure. Given $a \in I_n$ representing a class $\overline{a} \in I_n/I_{n+1}$ and $b \in I_m$ representing a class $\overline{b} \in I_m/I_{m+1}$, we define $\overline{a}\overline{b}$ to be the class in I_{n+m}/I_{n+m+1} represented by ab.

It is easy to check that if different choices of representing elements a, b were made in the above description, the value of $\overline{a}\overline{b}$ thus defined would still be the same, so that the definition is reasonable.

Example 2.7 Consider $R = \mathbb{Z}_{(p)}$ (the localization at (p)) with the (p)-adic topology. Then $\mathrm{gr}(R) = \mathbb{Z}/p[t]$, as a graded ring. For the successive quotients of ideals are of the form $\mathbb{Z}/p$, and it is easy to check that multiplication lines up in the appropriate form.

In general, as we will see below, when one takes the gr of a noetherian ring with the I-adic topology for some ideal I, one always gets a noetherian ring.

Definition 2.8 Let R be a filtered ring, and M a filtered R-module (with filtration $\{M_n\}$). We define the **associated graded module** $\mathrm{gr}(M)$ as the graded $\mathrm{gr}(R)$-module

$$\mathrm{gr}(M) = \bigoplus_n M_n/M_{n+1}$$

where multiplication by an element of $\mathrm{gr}(R)$ is defined in a similar manner as above.

In other words, we have defined a *functor* gr from the category of filtered R-modules to the category of *graded* $\mathrm{gr}(R)$ modules.

Let R be a filtered ring, and M a finitely generated filtered R-module. In general, $\mathrm{gr}(M)$ *cannot* be expected to be a finitely generated $\mathrm{gr}(R)$-module.

Example 2.9 Consider the ring $\mathbb{Z}_{(p)}$ (the localization of $\mathbb{Z}$ at p), which we endow with the p^2-adic (i.e., (p^2)-adic) filtration. The associated graded is $\mathbb{Z}/p^2[t]$.

Consider $M = \mathbb{Z}_{(p)}$ with the filtration $M_m = (p^m)$, i.e. the usual (p)-adic topology. The claim is that $\mathrm{gr}(M)$ is *not* a finitely generated $\mathbb{Z}/p^2[t]$-module. This will follow from **??** below, but we can see it directly: multiplication by t acts by zero on $\mathrm{gr}(M)$ (because this corresponds to multiplying by p^2 and shifting the degree by one). However, $\mathrm{gr}(M)$ is nonzero in every degree. If $\mathrm{gr}(M)$ were finitely generated, it would be a finitely generated $\mathbb{Z}/p^2\mathbb{Z}$-module, which it is not.

2.3 Topologies

We shall now see that filtered rings and modules come naturally with *topologies* on them.

Definition 2.10 A **topological ring** is a ring R together with a topology such that the natural maps

$$R \times R \to R, \quad (x, y) \mapsto x + y$$
$$R \times R \to R, \quad (x, y) \mapsto xy$$
$$R \to R, \quad x \mapsto -x$$

are continuous (where $R \times R$ has the product topology).

TO BE ADDED: discussion of algebraic objects in categories

In practice, the topological rings that we will be interested will exclusively be *linearly* topologized rings.

Definition 2.11 A topological ring is **linearly topologized** if there is a neighborhood basis at 0 consisting of open ideals.

Given a filtered ring R with a filtration of ideals $\{I_n\}$, we can naturally linearly topologize R. Namely, we take as a basis the cosets $x + I_n$ for $x \in R, n \in \mathbb{Z}_{\geq 0}$. It is then clear that the $\{I_n\}$ form a neighborhood basis at the origin (because any neighborhood $x + I_n$ containing 0 must just be I_n!).

Example 2.12 For instance, given any ring R and any ideal $I \subset R$, we can consider the **I-adic topology** on R. Here an element is "small" (i.e., close to zero) if it lies in a high power of I.

Proposition 2.13 *A topology on R defined by the filtration $\{I_n\}$ is Hausdorff if and only if $\bigcap I_n = 0$.*

Proof. Indeed, to say that R is Hausdorff is to say that any two distinct elements $x, y \in R$ can be separated by disjoint neighborhoods. If $\bigcap I_n = 0$, we can find N large such that $x - y \notin I_N$. Then $x + I_N, y + I_N$ are disjoint neighborhoods of x, y. The converse is similar: if $\bigcap I_n \neq 0$, then no neighborhoods can separate a nonzero element in $\bigcap I_n$ from 0. ▲

Similarly, if M is a filtered R-module with a filtration $\{M_n\}$, we can topologize M by choosing the $\{M_n\}$ to be a neighborhood basis at the origin. Then M becomes a *topological group,* that is a group with a topology such that the group operations are continuous. In the same way, we find:

Proposition 2.14 *The topology on M is Hausdorff if and only if $\bigcap M_n = 0$.*

Moreover, because of the requirement that $R_m M_n \subset M_{n+m}$, it is easy to see that the map

$$R \times M \to M$$

is itself continuous. Thus, M is a *topological* module.

Here is another example. Suppose M is a linearly topologized module with a basis of submodules $\{M_\alpha\}$ at the origin. Then any submodule $N \subset M$ becomes a linearly topologized module with a basis of submodules $\{N \cap M_\alpha\}$ at the origin with the relative topology.

Proposition 2.15 *Suppose M is filtered with the $\{M_n\}$. If $N \subset M$ is any submodule, then the closure $\overline{N}$ is the intersection $\bigcap N + M_n$.*

Proof. Recall that $x \in \overline{N}$ is the same as stipulating that every neighborhood of x intersect N. In other words, any basic neighborhood of x has to intersect N. This means that for each n, $x + M_n \cap N \neq \emptyset$, or in other words $x \in M_n + N$. ▲

§3 The Artin-Rees Lemma

We shall now show that for *noetherian* rings and modules, the I-adic topology is stable under passing to submodules; this useful result, the Artin-Rees lemma, will become indispensable in our analysis of dimension theory in the future.

More precisely, consider the following problem. Let R be a ring and $I \subset R$ an ideal. Then for any R-module M, we can endow M with the I-adic filtration $\{I^n M\}$, which defines a topology on M. If $N \subset M$ is a submodule, then N inherits the subspace topology from M (i.e. that defined by the filtration $\{I^n M \cap N\}$). But N can also be topologized by simply taking the I-adic topology on it. The Artin-Rees lemma states that these two approaches give the same result.

3.1 The Artin-Rees Lemma

Theorem 3.1 (Artin-Rees lemma) *Let R be noetherian, $I \subset R$ an ideal. Suppose M is a finitely generated R-module and $M' \subset M$ a submodule. Then the I-adic topology on M induces the I-adic topology on M'. More precisely, there is a constant c such that*

$$I^{n+c}M \cap M' \subset I^n M'.$$

So the two filtrations $\{I^n M \cap M'\}, \{I^n M'\}$ on M' are equivalent up to a shift.

Proof. The strategy to prove Artin-Rees will be as follows. Call a filtration $\{M_n\}$ on an R-module M (which is expected to be compatible with the I-adic filtration on R, i.e. $I^n M_m \subset M_{m+n}$ for all n, m) ***I*-good** if $IM_n = M_{n+1}$ for large $n \gg 0$. Right now, we have the very I-good filtration $\{I^n M\}$ on M, and the induced filtration $\{I^n M \cap M'\}$ on M'. The Artin-Rees lemma can be rephrased as saying that this filtration on M' is I-good: in fact, this is what we shall prove. It follows that if one has an I-good filtration on M, then the induced filtration on M' is itself I-good.

To do this, we shall give an interpretation of I-goodness in terms of the *blowup algebra*, and use its noetherianness. Recall that this is defined as $S = R \oplus I \oplus I^2 + \dots$, where multiplication is defined in the obvious manner (see Example 1.6). It can be regarded as a subring of the polynomial ring $R[t]$ where the coefficient of t^i is required to be in I^i. The blowup algebra is clearly a graded ring.

Given a filtration $\{M_n\}$ on an R-module M (compatible with the I-adic filtration of M), we can make $\bigoplus_{n=0}^{\infty} M_n$ into a *graded* S-module in an obvious manner.

Here is the promised interpretation of I-goodness:

Lemma 3.2 *Then the filtration $\{M_n\}$ of the finitely generated R-module M is I-good if and only if $\bigoplus M_n$ is a finitely generated S-module.*

Proof. Let $S_1 \subset S$ be the subset of elements of degree one. If $\bigoplus M_n$ is finitely generated as an S-module, then $S_1(\bigoplus M_n)$ and $\bigoplus M_n$ agree in large degrees by Lemma 1.20; however, this means that $IM_{n-1} = M_n$ for $n \gg 0$, which is I-goodness.

Conversely, if $\{M_n\}$ is an I-good filtration, then once the I-goodness starts (say, for $n > N$, we have $IM_n = M_{n+1}$), there is no need to add generators beyond M_N. In fact, we can use R-generators for $M_0, \dots, M_N$ in the appropriate degrees to generate $\bigoplus M_n$ as an R'-module. ▲

Finally, let $\{M_n\}$ be an I-good filtration on the finitely generated R-module M. Let $M' \subset M$ be a submodule; we will, as promised, show that the induced filtration on M' is I-good. Now the associated module $\bigoplus_{n=0}^{\infty}(I^n M \cap M')$ is an S-submodule of $\bigoplus_{n=0}^{\infty} M_n$, which by Lemma 3.2 is finitely generated. We will show next that S is noetherian, and consequently submodules of finitely generated modules are finitely generated. Applying Lemma 3.2 again, we will find that the induced filtration must be I-good.

Lemma 3.3 *Hypotheses as above, the blowup algebra R' is noetherian.*

Proof. Choose generators $x_1, \dots, x_n \in I$; then there is a map $R[y_1, \dots, y_n] \to S$ sending $y_i \to x_i$ (where x_i is in degree one). This is surjective. Hence by the basis theorem (**??**), R' is noetherian.▲

3.2 The Krull intersection theorem

We now prove a useful consequence of the Artin-Rees lemma and Nakayama's lemma. In fancier language, this states that the map from a noetherian local ring into its completion is an *embedding*. A priori, this might not be obvious. For instance, it might be surprising that the inverse limit of the highly torsion groups $\mathbb{Z}/p^n$ turns out to be the torsion-free ring of p-adic integers.

Theorem 3.4 (Krull intersection theorem) *Let R be a local noetherian ring with maximal ideal $\mathfrak{m}$. Then,*

$$\bigcap \mathfrak{m}^i = (0).$$

Proof. Indeed, the $\mathfrak{m}$-adic topology on $\bigcap \mathfrak{m}^i$ is the restriction of the $\mathfrak{m}$-adic topology of R on $\bigcap \mathfrak{m}^i$ by the Artin-Rees lemma (Theorem 3.1). However, $\bigcap \mathfrak{m}^i$ is contained in every $\mathfrak{m}$-adic neighborhood of 0 in R; the induced topology on $\bigcap \mathfrak{m}^i$ is thus the indiscrete topology.

But to say that the $\mathfrak{m}$-adic topology on a module N is indiscrete is to say that $\mathfrak{m}N = N$, so $N = 0$ by Nakayama. The result is thus clear.

By similar logic, or by localizing at each maximal ideal, we find:

Corollary 3.5 *If R is a commutative ring and I is contained in the Jacobson radical of R, then $\bigcap I^n = 0$.*

It turns out that the Krull intersection theorem can be proved in the following elementary manner, due to Perdry in [Per04]. The argument does not use the Artin-Rees lemma. One can prove:

Theorem 3.6 ([Per04]) *Suppose R is a noetherian ring, $I \subset R$ an ideal. Suppose $b \in \bigcap I^n$. Then as ideals $(b) = (b)I$.*

In particular, it follows easily that $\bigcap I^n = 0$ under either of the following conditions:

1. I is contained in the Jacobson radical of R.
2. R is a domain and I is proper.

Proof. Let $a_1, \dots, a_k \in I$ be generators. For each n, the ideal I^n consists of the values of all homogeneous polynomials in $R[x_1, \dots, x_k]$ of degree n evaluated on the tuple $(a_1, \dots, a_k)$, as one may easily see.

It follows that if $b \in \bigcap I^n$, then for each n there is a polynomial $P_n \in R[x_1, \dots, x_k]$ which is homogeneous of degree n and which satisfies

$$P_n(a_1, \dots, a_k) = b.$$

The ideal generated by all the P_n in $R[x_1, \dots, x_k]$ is finitely generated by the Hilbert basis theorem. Thus there is N such that

$$P_N = Q_1 P_1 + Q_2 P_2 + \cdots + Q_{N-1} P_{N-1}$$

for some polynomials $Q_i \in R[x_1, \dots, x_k]$. By taking homogeneous components, we can assume moreover that Q_i is homogeneous of degree $N - i$ for each i. If we evaluate each at $(a_1, \dots, a_k)$ we find

$$b = b(Q_1(a_1, \dots, a_k) + \cdots + Q_{N-1}(a_1, \dots, a_k)).$$

But the $Q_i(a_1, \dots, a_k)$ lie in I as all the a_i do and Q_i is homogeneous of positive degree. Thus b equals b times something in I. ▲

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