Contents

Chapter 10
Dimension theory

Dimension theory assigns to each commutative ring—say, noetherian—an invariant called the dimension. The most standard definition, that of Krull dimension (which we shall not adopt at first), defines the dimension in terms of the maximal lengths of ascending chains of prime ideals. In general, however, the geometric intuition behind dimension is that it should assign to an affine ring—say, one of the form $\mathbb{C}[x_1, \dots, X_n]/I$—something like the "topological dimension" of the affine variety in $\mathbb{C}^n$ cut out by the ideal I.

In this chapter, we shall obtain three different expressions for the dimension of a noetherian local ring $(R, \mathfrak{m})$, each of which will be useful at different times in proving results.

§1 The Hilbert function and the dimension of a local ring

1.1 Integer-valued polynomials

It is now necessary to do a small amount of general algebra.

Let $P \in \mathbb{Q}[t]$. We consider the question of when P maps the integers $\mathbb{Z}$, or more generally the sufficiently large integers, into $\mathbb{Z}$. Of course, any polynomial in $\mathbb{Z}[t]$ will do this, but there are others: consider $\frac{1}{2}(t^2 - t)$, for instance.

Proposition 1.1 *Let $P \in \mathbb{Q}[t]$. Then $P(m)$ is an integer for $m \gg 0$ integral if and only if P can be written in the form*

$$P(t) = \sum_n c_n \binom{t}{n}, \quad c_n \in \mathbb{Z}.$$

In particular, $P(\mathbb{Z}) \subset \mathbb{Z}$.

So P is a $\mathbb{Z}$-linear function of binomial coefficients.

Proof. Note that the set $\left\{\binom{t}{n}\right\}_{n \in \mathbb{Z}_{\geq 0}}$ forms a basis for the set of polynomials $\mathbb{Q}[t]$. It is thus clear that $P(t)$ can be written as a rational combination $\sum c_n \binom{t}{n}$ for the $c_n \in \mathbb{Q}$. We need to argue that the $c_n \in \mathbb{Z}$ in fact.

Consider the operator Δ defined on functions $\mathbb{Z} \to \mathbb{C}$ as follows:

$$(\Delta f)(m) = f(m) - f(m-1).$$

It is obvious that if f takes integer values for $m \gg 0$, then so does Δf. It is also easy to check that $\Delta \binom{t}{n} = \binom{t}{n-1}$.

By looking at the function $\Delta P = \sum c_n \binom{t}{n-1}$ (which takes values in $\mathbb{Z}$), it is easy to see that the $c_n \in \mathbb{Z}$ by induction on the degree. It is also easy to see directly that the binomial coefficients take values in $\mathbb{Z}$ at *all* arguments. ▲

1.2 Definition and examples

Let R be a ring.

Question What is a good definition for $\dim(R)$? Actually, more generally, what is the dimension of R at a given "point" (i.e. prime ideal)?

Geometrically, think of $\operatorname{Spec} R$, for any ring; pick some point corresponding to a maximal ideal $\mathfrak{m} \subset R$. We want to define the **dimension of** R at $\mathfrak{m}$. This is to be thought of kind of like "dimension over the complex numbers," for algebraic varieties defined over $\mathbb{C}$. But it should be purely algebraic. What might you do?

Here is an idea. For a topological space X to be n-dimensional at $x \in X$, there should be n coordinates at the point x. In other words, the point x should be uniquely defined by the zero locus of n points on the space. Motivated by this, we could try defining $\dim_{\mathfrak{m}} R$ to be the number of generators of $\mathfrak{m}$. However, this is a bad definition, as $\mathfrak{m}$ may not have the same number of generators as $\mathfrak{m}R_{\mathfrak{m}}$. In other words, it is not a truly *local* definition.

Example 1.2 Let R be a noetherian integrally closed domain which is not a UFD. Let $\mathfrak{p} \subset R$ be a prime ideal which is minimal over a principal ideal but which is not itself principal. Then $\mathfrak{p}R_{\mathfrak{p}}$ is generated by one element, as we will eventually see, but $\mathfrak{p}$ is not.

We want our definition of dimension to be local. So this leads us to:

Definition 1.3 If R is a (noetherian) *local* ring with maximal ideal $\mathfrak{m}$, then the **embedding dimension** of R, denoted $\operatorname{Emdim} R$ is the minimal number of generators for $\mathfrak{m}$. If R is a noetherian ring and $\mathfrak{p} \subset R$ a prime ideal, then the **embedding dimension at** $\mathfrak{p}$ is that of the local ring $R_{\mathfrak{p}}$.

In the above definition, it is clearly sufficient to study what happens for local rings, and we impose that restriction for now. By Nakayama's lemma, the embedding dimension is the minimal number of generators of $\mathfrak{m}/\mathfrak{m}^2$, or the $R/\mathfrak{m}$-dimension of that vector space:

$$\operatorname{Emdim} R = \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2.$$

In general, however, the embedding dimension is not going to coincide with the intuitive "geometric" dimension of an algebraic variety.

Example 1.4 Let $R = \mathbb{C}[t^2, t^3] \subset \mathbb{C}[t]$, which is the coordinate ring of a cubic curve $y^2 = x^3$ as $R \simeq \mathbb{C}[x, y]/(x^2 - y^3)$ via $x = t^3, y = t^2$. Let us localize at the prime ideal $\mathfrak{p} = (t^2, t^3)$: we get $R_{\mathfrak{p}}$.

Now $\operatorname{Spec} R$ is singular at the origin. In fact, as a result, $\mathfrak{p}R_{\mathfrak{p}} \subset R_{\mathfrak{p}}$ needs two generators, but the variety it corresponds to is one-dimensional.

So the embedding dimension is the smallest dimension into which you can embed R into a smooth space. But for singular varieties this is not the dimension we want.

So instead of considering simply $\mathfrak{m}/\mathfrak{m}^2$, let us consider the *sequence* of finite-dimensional vector spaces

$$\mathfrak{m}^k/\mathfrak{m}^{k+1}.$$

Computing these dimensions as a function of k gives some invariant that describes the local geometry of $\operatorname{Spec} R$.

We shall eventually prove:

Theorem 1.5 *Let $(R, \mathfrak{m})$ be a local noetherian ring. Then there exists a polynomial $f \in \mathbb{Q}[t]$ such that*

$$f(n) = \ell(R/\mathfrak{m}^n) = \sum_{i=0}^{n-1} \dim \mathfrak{m}^i/\mathfrak{m}^{i+1} \quad \forall n \gg 0.$$

Moreover, $\deg f \leq \dim \mathfrak{m}/\mathfrak{m}^2$.

Note that this polynomial is well-defined, as any two polynomials agreeing for large n coincide. Note also that $R/\mathfrak{m}^n$ is artinian so of finite length, and that we have used the fact that the length is additive for short exact sequences. We would have liked to write $\dim R/\mathfrak{m}^n$, but we can't, in general, so we use the substitute of the length.

Based on this, we define:

Definition 1.6 The **dimension** of the local ring R is the degree of the polynomial f above. For an arbitrary noetherian ring R, we define $\dim R = \sup_{\mathfrak{p} \in \mathrm{Spec}\, R} \dim(R_{\mathfrak{p}})$.

Let us now do a few example computations.

Example 1.7 (The affine line) Consider the local ring $(R, \mathfrak{m}) = \mathbb{C}[t]_{(t)}$. Then $\mathfrak{m} = (t)$ and $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is one-dimensional, generated by t^k. In particular, the ring has dimension one.

Example 1.8 (A singular curve) Consider $R = \mathbb{C}[t^2, t^3]_{(t^2,t^3)}$, the local ring of $y^2 = x^3$ at zero. Then $\mathfrak{m}^n$ is generated by $t^{2n}, t^{2n+1}, \dots$. $\mathfrak{m}^{n+1}$ is generated by $t^{2n+2}, t^{2n+3}, \dots$. So the quotients all have dimension two. The dimension of these quotients is a little larger than in Example 1.7, but they do not grow. The ring still has dimension one.

Example 1.9 (The affine plane) Consider $R = \mathbb{C}[x, y]_{(x,y)}$. Then $\mathfrak{m}^k$ is generated by polynomials in x, y that are homogeneous in degree k. So $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ has dimensions that *grow* linearly in k. This is a genuinely two-dimensional example.

It is this difference between constant linear and quadratic growth in $R/\mathfrak{m}^n$ as $n \to \infty$, and not the size of the initial terms, that we want for our definition of dimension.

Let us now generalize Example 1.7 and Example 1.9 above to affine spaces of arbitrary dimension.

Example 1.10 (Affine space) Consider $R = \mathbb{C}[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$. This represents the variety $\mathbb{C}^n = \mathbb{A}^n_{\mathbb{C}}$ near the origin geometrically, so it should intuitively have dimension n. Let us check that it does.

Namely, we need to compute the polynomial f above. Here $R/\mathfrak{m}^k$ looks like the set of polynomials of degree $< k$ over $\mathbb{C}$. The dimension as a vector space of this is given by some binomial coefficient $\binom{n+k-1}{n}$. This is a polynomial in k of degree n. In particular, $\ell(R/\mathfrak{m}^k)$ grows like k^n. So R is n-dimensional.

Finally, we offer one more example, showing that DVRs have dimension one. In fact, among noetherian integrally closed local domains, DVRs are *characterized* by this property (**??** of **??**).

Example 1.11 (The dimension of a DVR) Let R be a DVR. Then $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is of length one for each k. So $R/\mathfrak{m}^k$ has length k. Thus we can take $f(t) = t$, so R has dimension one.

1.3 The Hilbert function is a polynomial

While we have given a definition of dimension and computed various examples, we have yet to check that our definition is well-defined. Namely, we have to prove Theorem 1.5.

Proof (Proof of Theorem 1.5). Fix a noetherian local ring $(R, \mathfrak{m})$. We are to show that $\ell(R/\mathfrak{m}^n)$ is a polynomial for $n \gg 0$. We also have to bound this degree by $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$, the embedding dimension. We will do this by reducing to a general fact about graded modules over a polynomial ring.

Let $S = \bigoplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Then S has a natural grading, and in fact it is a graded ring in a natural way from the multiplication map

$$\mathfrak{m}^{n_1} \times \mathfrak{m}^{n_2} \to \mathfrak{m}^{n_1+n_2}.$$

In fact, S is the *associated graded ring* of the $\mathfrak{m}$-adic filtration. Note that $S_0 = R/\mathfrak{m}$ is a field, which we will denote by k. So S is a graded k-algebra.

Lemma 1.12 *S is a finitely generated k-algebra. In fact, S can be generated by at most* $\mathrm{Emdim}(R)$ *elements.*

Proof. Let $x_1, \dots, x_r$ be generators for $\mathfrak{m}$ with $r = \mathrm{Emdim}(R)$. They (or rather, their images) are thus a k-basis for $\mathfrak{m}/\mathfrak{m}^2$. Then their images in $\mathfrak{m}/\mathfrak{m}^2 \subset S$ generate S. This follows because S_1 generates S as an S_0-algebra: the products of the elements in $\mathfrak{m}$ generate the higher powers of $\mathfrak{m}$.▲

So S is a graded quotient of the polynomial ring $k[t_1, \dots, t_r]$, with t_i mapping to x_i. In particular, S is a finitely generated, graded $k[t_1, \dots, t_r]$-module. Note that also $\ell(R/\mathfrak{m}^n) = \dim_k(S_0) + \dots + \dim_k(S_{n-1})$ for any n, thanks to the filtration. This is the invariant we are interested in.

It will now suffice to prove the following more general proposition.

Proposition 1.13 *Let M be any finitely generated graded module over the polynomial ring $k[x_1, \dots, x_r]$. Then there exists a polynomial $f_M^+ \in \mathbb{Q}[t]$ of degree $\leq r$, such that*

$$f_M^+(t) = \sum_{s \leq t} \dim M_s \quad t \gg 0.$$

Applying this to $M = S$ will give the desired result. We can forget about everything else, and look at this problem over graded polynomial rings.

This function is called the **Hilbert function**.

Proof (Proof of Proposition 1.13). Note that if we have an exact sequence of graded modules over the polynomial ring,

$$0 \to M' \to M \to M'' \to 0,$$

and polynomials $f_{M'}, f_{M''}$ as in the proposition, then f_M exists and

$$f_M = f_{M'} + f_{M''}.$$

This is obvious from the definitions. Next, we observe that if M is a finitely generated graded module, over two different polynomial rings, but with the same grading, then the existence (and value) of f_M is independent of which polynomial ring one considers. Finally, we observe that it is sufficient to prove that $f_M(t) = \dim M_t$ is a polynomial in t for $t \gg 0$.

We will use these three observations and induct on n.

If $n = 0$, then M is a finite-dimensional graded vector space over k, and the grading must be concentrated in finitely many degrees. Thus the result is evident as $f_M(t)$ will just equal $\dim M$ (which will be the appropriate dimension for $t \gg 0$).

Suppose $n > 0$. Then consider the filtration of M

$$0 \subset \ker(x_1 : M \to M) \subset \ker(x_1^2 : M \to M) \subset \dots \subset M.$$

This must stabilize by noetherianness at some $M' \subset M$. Each of the quotients $\ker(x_1^i)/\ker(x_1^{i+1})$ is a finitely generated module over $k[x_1, \dots, x_n]/(x_1)$, which is a smaller polynomial ring. So each of these quotients $\ker(x_1^{i+1})/\ker(x_1^i)$ has a Hilbert function of degree $\leq n-1$ by the inductive hypothesis.

Climbing up the filtration, we see that M' has a Hilbert function which is the sum of the Hilbert functions of these quotients $\ker(x_1^{i+1})/\ker(x_1^i)$. In particular, $f_{M'}$ exists. If we show that $f_{M/M'}$ exists, then f_M necessarily exists. So we might as well show that the Hilbert function f_M exists when x_1 is a non-zerodivisor on M.

So, we have reduced to the case where $M \stackrel{x_1}{\to} M$ is injective. Now M has a filtration

$$M \supset x_1 M \supset x_1^2 M \supset \dots$$

which is an exhaustive filtration of M in that nothing can be divisible by powers of x_1 over and over, or the degree would not be finite. So it follows that $\bigcap x_1^m M = 0$.

Let $N = M/x_1 M$, which is isomorphic to $x_1^m M/x_1^{m+1} M$ since $M \stackrel{x_1}{\to} M$ is injective. Here N is a finitely generated graded module over $k[x_2, \dots, x_n]$, and by the inductive hypothesis on n, we see that there is a polynomial f_N^+ of degree $\leq n-1$ such that

$$f_N^+(t) = \sum_{t' \leq t} \dim N_{t'}, \quad t \gg 0.$$

Fix $t \gg 0$ and consider the k-vector space M_t, which has a finite filtration

$$M_t \supset (x_1 M)_t \supset (x_1^2 M)_t \supset \dots$$

which has successive quotients that are the graded pieces of $N \simeq M/x_1 M \simeq x_1 M/x_1^2 M \simeq \dots$ in dimensions $t, t-1, \dots$. We find that

$$(x_1^2 M)_t/(x_1^3 M)_t \simeq N_{t-2},$$

for instance. Summing this, we find that

$$\dim M_t = \dim N_t + \dim N_{t-1} + \dots.$$

The sum above is actually finite. In fact, by finite generation, there is $K \gg 0$ such that $\dim N_q = 0$ for $q < -K$. From this, we find that

$$\dim M_t = \sum_{t'=-K}^{t} \dim N_{t'},$$ ▲

which implies that $\dim M_t$ is a polynomial for $t \gg 0$. This completes the proof. ▲

Let $(R, \mathfrak{m})$ a noetherian local ring and M a finitely generated R-module.

Proposition 1.14 *$\ell(M/\mathfrak{m}^m M)$ is a polynomial for $m \gg 0$.*

Proof. This follows from Proposition 1.13, and in fact we have essentially seen the argument above. Indeed, we consider the associated graded module

$$N = \bigoplus \mathfrak{m}^k M/\mathfrak{m}^{k+1} M,$$

which is finitely generated over the associated graded ring

$$\bigoplus \mathfrak{m}^k/\mathfrak{m}^{k+1}.$$ ▲

Consequently, the graded pieces of N have dimensions growing polynomially for large degrees. This implies the result.

Definition 1.15 We define the **Hilbert function** $H_M(m)$ to be the unique polynomial such that

$$H_M(m) = \ell(M/\mathfrak{m}^m M), \quad m \gg 0.$$

It is clear, incidentally, that H_M is integer-valued, so we see by Proposition 1.1 that H_M is a $\mathbb{Z}$-linear combination of binomial coefficients.

1.4 The dimension of a module

Let R be a local noetherian ring with maximal ideal $\mathfrak{m}$. We have seen (Proposition 1.14) that there is a polynomial $H(t)$ with

$$H(t) = \ell(R/\mathfrak{m}^t), \quad t \gg 0.$$

Earlier, we defined the **dimension** of R is the degree of f_M^+. Since the degree of the Hilbert function is at most the number of generators of the polynomial ring, we saw that

$$\dim R \leq \operatorname{Emdim} R.$$

Armed with the machinery of the Hilbert function, we can extend this definition to modules.

Definition 1.16 If R is local noetherian, and N a finite R-module, then N has a Hilbert polynomial $H_N(t)$ which when evaluated at $t \gg 0$ gives the length $\ell(N/\mathfrak{m}^t N)$. We say that the **dimension of** N is the degree of this Hilbert polynomial.

Clearly, the dimension of the *ring* R is the same thing as that of the *module* R.

We next show that the dimension behaves well with respect to short exact sequences. This is actually slightly subtle since, in general, tensoring with $R/\mathfrak{m}^t$ is not exact; it turns out to be *close* to being exact by the Artin-Rees lemma. On the other hand, the corresponding fact for modules over a *polynomial ring* is very easy, as no tensoring was involved in the definition.

Proposition 1.17 *Suppose we have an exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

of graded modules over a polynomial ring $k[x_1, \dots, x_n]$. *Then*

$$f_M(t) = f_{M'}(t) + f_{M''}(t), \quad f_M^+(t) = f_{M'}^+(t) + f_{M''}^+(t).$$

As a result, $\deg f_M = \max \deg f_{M'}, \deg f_{M''}$.

Proof. The first part is obvious as the dimension is additive on vector spaces. The second part follows because Hilbert functions have nonnegative leading coefficients. ▲

Proposition 1.18 *Fix an exact sequence*

$$0 \to N' \to N \to N'' \to 0$$

of finite R-modules. Then $\dim N = \max(\dim N', \dim N'')$.

Proof. We have an exact sequence

$$0 \to K \to N/\mathfrak{m}^t N \to N''/\mathfrak{m}^t N'' \to 0$$

where K is the kernel. Here $K = (N' + \mathfrak{m}^t N)/\mathfrak{m}^t N = N'/(N' \cap \mathfrak{m}^t N)$. This is not quite $N'/\mathfrak{m}^t N'$, but it's pretty close. We have a surjection

$$N'/\mathfrak{m}^t N \twoheadrightarrow N'/(N' \cap \mathfrak{m}^t N) = K.$$

In particular,

$$\ell(K) \leq \ell(N'/\mathfrak{m}^t N').$$

On the other hand, we have the Artin-Rees lemma, which gives an inequality in the opposite direction. We have a containment

$$\mathfrak{m}^t N' \subset N' \cap \mathfrak{m}^t N \subset \mathfrak{m}^{t-c} N'$$

for some c. This implies that $\ell(K) \geq \ell(N'/\mathfrak{m}^{t-c}N')$.

Define $M = \bigoplus \mathfrak{m}^t N/\mathfrak{m}^{t+1}N$, and define M', M'' similarly in terms of N', N''. Then we have seen that

$$\boxed{f^+_M(t-c) \leq \ell(K) \leq f^+_M(t).}$$

We also know that the length of K plus the length of $N''/\mathfrak{m}^t N''$ is $f^+_M(t)$, i.e.

$$\ell(K) + f^+_{M''}(t) = f^+_M(t).$$

Now the length of K is a polynomial in t which is pretty similar to $f^+_{M'}$, in that the leading coefficient is the same. So we have an approximate equality $f^+_{M'}(t) + f^+_{M''}(t) \simeq f^+_M(t)$. This implies the result since the degree of f^+_M is $\dim N$ (and similarly for the others). ▲

Proposition 1.19 $\dim R$ *is the same as* $\dim R/\operatorname{Rad} R$.

I.e., the dimension doesn't change when you kill off nilpotent elements, which is what you would expect, as nilpotents don't affect $\operatorname{Spec}(R)$.

Proof. For this, we need a little more information about Hilbert functions. We thus digress substantially.

Finally, let us return to the claim about dimension and nilpotents. Let R be a local noetherian ring and $I = \operatorname{Rad}(R)$. Then I is a finite R-module. In particular, I is nilpotent, so $I^n = 0$ for $n \gg 0$. We will show that

$$\dim R/I = \dim R/I^2 = \dots$$

which will imply the result, as eventually the powers become zero.

In particular, we have to show for each k,

$$\dim R/I^k = \dim R/I^{k+1}.$$

There is an exact sequence

$$0 \to I^k/I^{k+1} \to R/I^{k+1} \to R/I^k \to 0.$$

The dimension of these rings is the same thing as the dimensions as R-modules. So we can use this short exact sequence of modules. By the previous result, we are reduced to showing that

$$\dim I^k/I^{k+1} \leq \dim R/I^k.$$

Well, note that I kills I^k/I^{k+1}. In particular, I^k/I^{k+1} is a finitely generated R/I^k-module. There is an exact sequence

$$\bigoplus_N R/I^k \to I^k/I^{k+1} \to 0$$

which implies that $\dim I^k/I^{k+1} \leq \dim \bigoplus_N R/I^k = \dim R/I^k$. ▲

Example 1.20 Let $\mathfrak{p} \subset \mathbb{C}[x_1, \dots, x_n]$ and let $R = (\mathbb{C}[x_1, \dots, x_n]/\mathfrak{p})_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$. What is $\dim R$? What does dimension mean for coordinate rings over $\mathbb{C}$?

Recall by the Noether normalization theorem that there exists a polynomial ring $\mathbb{C}[y_1, \dots, y_m]$ contained in $S = \mathbb{C}[x_1, \dots, x_n]/\mathfrak{p}$ and S is a finite integral extension over this polynomial ring. We claim that

$$\dim R = m.$$

There is not sufficient time for that today.

1.5 Dimension depends only on the support

Let $(R, \mathfrak{m})$ be a local noetherian ring. Let M be a finitely generated R-module. We defined the **Hilbert polynomial** of M to be the polynomial which evaluates at $t \gg 0$ to $\ell(M/\mathfrak{m}^t M)$. We proved last time that such a polynomial always exists, and called its degree the **dimension of** M. However, we shall now see that $\dim M$ really depends only on the support[1] $\operatorname{supp} M$. In this sense, the dimension is really a statement about the *topological space* $\operatorname{supp} M \subset \operatorname{Spec} R$, not about M itself.

In other words, we will prove:

Proposition 1.21 $\dim M$ *depends only on* $\operatorname{supp} M$*.*

In fact, we shall show:

Proposition 1.22 $\dim M = \max_{\mathfrak{p} \in \operatorname{supp} M} \dim R/\mathfrak{p}$.

Proof. By Proposition 2.9 in Chapter 5, there is a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M,$$

such that each of the successive quotients is isomorphic to $R/\mathfrak{p}_i \subset R$ for some prime ideal $\mathfrak{p}_i$. Given a short exact sequence of modules, we know that the dimension in the middle is the maximum of the dimensions at the two ends (Proposition 1.18). Iterating this, we see that the dimension of M is the maximum of the dimension of the successive quotients M_i/M_{i-1}.

But the $\mathfrak{p}_i$'s that occur are all in $\operatorname{supp} M$, so we find

$$\dim M = \max_{\mathfrak{p}_i} R/\mathfrak{p}_i \leq \max_{\mathfrak{p} \in \operatorname{supp} M} \dim R/\mathfrak{p}.$$

We must show the reverse inequality. But fix any prime $\mathfrak{p} \in \operatorname{supp} M$. Then $M_{\mathfrak{p}} \neq 0$, so one of the $R/\mathfrak{p}_i$ localized at $\mathfrak{p}$ must be nonzero, as localization is an exact functor. Thus $\mathfrak{p}$ must contain some $\mathfrak{p}_i$. So $R/\mathfrak{p}$ is a quotient of $R/\mathfrak{p}_i$. In particular,

$$\dim R/\mathfrak{p} \leq \dim R/\mathfrak{p}_i.$$

▲

Having proved this, we throw out the notation $\dim M$, and henceforth write instead $\dim \operatorname{supp} M$.

Example 1.23 Let $R' = \mathbb{C}[x_1, \dots, x_n]/\mathfrak{p}$. Noether normalization says that there exists a finite injective map $\mathbb{C}[y_1, \dots, y_a] \to R'$. The claim is that

$$\dim R'_{\mathfrak{m}} = a$$

for any maximal ideal $\mathfrak{m} \subset R'$. We are set up to prove a slightly weaker definition. In particular (see below for the definition of the dimension of a non-local ring), by the proposition, we find the weaker claim

$$\dim R' = a,$$

as the dimension of a polynomial ring $\mathbb{C}[y_1, \dots, y_a]$ is a. (**I don't think we have proved this yet.**)

[1] Recall that $\operatorname{supp} M = \{\mathfrak{p} : M_{\mathfrak{p}} \neq 0\}$.

§2 Other definitions and characterizations of dimension

2.1 The topological characterization of dimension

We now want a topological characterization of dimension. So, first, we want to study how dimension changes as we do things to a module. Let M be a finitely generated R-module over a local noetherian ring R. Let $x \in \mathfrak{m}$ for $\mathfrak{m}$ as the maximal ideal. You might ask

> What is the relation between $\dim \operatorname{supp} M$ and $\dim \operatorname{supp} M/xM$?

Well, M surjects onto M/xM, so we have the inequality $\geq$. But we think of dimension as describing the number of parameters you need to describe something. The number of parameters shouldn't change too much with going from M to M/xM. Indeed, as one can check,

$$\operatorname{supp} M/xM = \operatorname{supp} M \cap V(x)$$

and intersecting $\operatorname{supp} M$ with the "hypersurface" $V(x)$ should shrink the dimension by one.

We thus make:

Prediction

$$\dim \operatorname{supp} M/xM = \dim \operatorname{supp} M - 1.$$

Obviously this is not always true, e.g. if x acts by zero on M. But we want to rule that out. Under reasonable cases, in fact, the prediction is correct:

Proposition 2.1 *Suppose $x \in \mathfrak{m}$ is a nonzerodivisor on M. Then*

$$\dim \operatorname{supp} M/xM = \dim \operatorname{supp} M - 1.$$

Proof. To see this, we look at Hilbert polynomials. Let us consider the exact sequence

$$0 \to xM \to M \to M/xM \to 0$$

which leads to an exact sequence for each t,

$$0 \to xM/(xM \cap \mathfrak{m}^t M) \to M/\mathfrak{m}^t M \to M/(xM + \mathfrak{m}^t M) \to 0.$$

For t large, the lengths of these things are given by Hilbert polynomials, as the thing on the right is $M/xM \otimes_R R/\mathfrak{m}^t$. We have

$$f_M^+(t) = f_{M/xM}^+(t) + \ell(xM/(xM \cap \mathfrak{m}^t M), \quad t \gg 0.$$

In particular, $\ell(xM/(xM \cap \mathfrak{m}^t M))$ is a polynomial in t. What can we say about it? Well, $xM \simeq M$ as x is a nonzerodivisor. In particular

$$xM/(xM \cap \mathfrak{m}^t M) \simeq M/N_t$$

where

$$N_t = \left\{ a \in M : xa \in \mathfrak{m}^t M \right\}.$$

In particular, $N_t \supset \mathfrak{m}^{t-1} M$. This tells us that $\ell(M/N_t) \leq \ell(M/\mathfrak{m}^{t-1} M) = f_M^+(t-1)$ for $t \gg 0$. Combining this with the above information, we learn that

$$f_M^+(t) \leq f_{M/xM}^+(t) + f_M^+(t-1),$$

which implies that $f^+_{M/xM}(t)$ is at least the successive difference $f^+_M(t) - f^+_M(t-1)$. This last polynomial has degree $\dim \operatorname{supp} M - 1$. In particular, $f^+_{M/xM}(t)$ has degree at least $\dim \operatorname{supp} M - 1$. This gives us one direction, actually the hard one. We showed that intersecting something with codimension one doesn't drive the dimension down too much.

Let us now do the other direction. We essentially did this last time via the Artin-Rees lemma. We know that $N_t = \{a \in M : xa \in \mathfrak{m}^t\}$. The Artin-Rees lemma tells us that there is a constant c such that $N_{t+c} \subset \mathfrak{m}^t M$ for all t. Therefore, $\ell(M/N_{t+c}) \geq \ell(M/\mathfrak{m}^t M) = f^+_M(t), t \gg 0$. Now remember the exact sequence $0 \to M/N_t \to M/\mathfrak{m}^t M \to M/(xM + \mathfrak{m}^t M) \to 0$. We see from this that

$$\ell(M/\mathfrak{m}^t M) = \ell(M/N_t) + f^+_{M/xM}(t) \geq f^+_M(t-c) + f^+_{M/xM}(t), \quad t \gg 0,$$

which implies that

$$f^+_{M/xM}(t) \leq f^+_M(t) - f^+_M(t-c),$$

so the degree must go down. And we find that $\deg f^+_{M/xM} < \deg f^+_M$. ▲

This gives us an algorithm of computing the dimension of an R-module M. First, it reduces to computing $\dim R/\mathfrak{p}$ for $\mathfrak{p} \subset R$ a prime ideal. We may assume that R is a domain and that we are looking for $\dim R$. Geometrically, this corresponds to taking an irreducible component of $\operatorname{Spec} R$.

Now choose any $x \in R$ such that x is nonzero but noninvertible. If there is no such element, then R is a field and has dimension zero. Then compute $\dim R/x$ (recursively) and add one.

Notice that this algorithm said nothing about Hilbert polynomials, and only talked about the structure of prime ideals.

2.2 Recap

Last time, we were talking about dimension theory. Recall that R is a local noetherian ring with maximal ideal $\mathfrak{m}$, M a finitely generated R-module. We can look at the lengths $\ell(M/\mathfrak{m}^t M)$ for varying t; for $t \gg 0$ this is a polynomial function. The degree of this polynomial is called the **dimension** of $\operatorname{supp} M$.

Remark If $M = 0$, then we define $\dim \operatorname{supp} M = -1$ by convention.

Last time, we showed that if $M \neq 0$ and $x \in \mathfrak{m}$ such that x is a nonzerodivisor on M (i.e. $M \xrightarrow{x} M$ injective), then

$$\boxed{\dim \operatorname{supp} M/xM = \dim \operatorname{supp} M - 1.}$$

Using this, we could give a recursion for calculating the dimension. To compute $\dim R = \dim \operatorname{Spec} R$, we note three properties:

1. $\dim R = \sup_{\mathfrak{p} \text{ a minimal prime}} R/\mathfrak{p}$. Intuitively, this says that a variety which is the union of irreducible components has dimension equal to the maximum of these irreducibles.
2. $\dim R = 0$ for R a field. This is obvious from the definitions.
3. If R is a domain, and $x \in \mathfrak{m} - \{0\}$, then $\dim R/(x) + 1 = \dim R$. This is obvious from the boxed formula as x is a nonzerodivisor.

These three properties *uniquely characterize* the dimension invariant.

More precisely, if $d : \{\text{local noetherian rings}\} \to \mathbb{Z}_{\geq 0}$ **satisfies the above three properties, then** $d = \dim$.

Proof. Induction on $\dim R$. It is clearly sufficient to prove this for R a domain. If R is a field, then it's clear; if $\dim R > 0$, the third condition lets us reduce to a case covered by the inductive hypothesis (i.e. go down). ▲

Let us rephrase 3 above:

3': If R is a domain and not a field, then

$$\dim R = \sup_{x \in \mathfrak{m} - 0} \dim R/(x) + 1.$$

Obviously 3' implies 3, and it is clear by the same argument that 1,2, 3' characterize the notion of dimension.

2.3 Krull dimension

We shall now define another notion of dimension, and show that it is equivalent to the older one by showing that it satisfies these axioms.

Definition 2.2 Let R be a commutative ring. A **chain of prime ideals** in R is a finite sequence

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n.$$

This chain is said to have **length** n**.**

Definition 2.3 The **Krull dimension** of R is equal to the maximum length of any chain of prime ideals. This might be ∞, but we will soon see this cannot happen for R local and noetherian.

Remark For any maximal chain $\{\mathfrak{p}_i, 0 \leq i \leq n\}$ of primes (i.e. which can't be expanded), we must have that $\mathfrak{p}_0$ is minimal prime and $\mathfrak{p}_n$ a maximal ideal.

Theorem 2.4 *For a noetherian local ring R, the Krull dimension of R exists and is equal to the usual* $\dim R$*.*

Proof. We will show that the Krull dimension satisfies the above axioms. For now, write Krdim for Krull dimension.

1. First, note that $\mathrm{Krdim}(R) = \max_{\mathfrak{p} \in R \text{ minimal}} \mathrm{Krdim}(R/\mathfrak{p})$. This is because any chain of prime ideals in R contains a minimal prime. So any chain of prime ideals in R can be viewed as a chain in *some* $R/\mathfrak{p}$, and conversely.

2. Second, we need to check that $\mathrm{Krdim}(R) = 0$ for R a field. This is obvious, as there is precisely one prime ideal.

3. The third condition is interesting. We must check that for $(R, \mathfrak{m})$ a local domain,

   $$\mathrm{Krdim}(R) = \max_{x \in \mathfrak{m} - \{0\}} \mathrm{Krdim}(R/(x)) + 1.$$

   If we prove this, we will have shown that condition 3' is satisfied by the Krull dimension. It will follow by the inductive argument above that $\mathrm{Krdim}(R) = \dim(R)$ for any R. There are two inequalities to prove. First, we must show

   $$\mathrm{Krdim}(R) \geq \mathrm{Krdim}(R/x) + 1, \quad \forall x \in \mathfrak{m} - 0.$$

   So suppose $k = \mathrm{Krdim}(R/x)$. We want to show that there is a chain of prime ideals of length $k+1$ in R. So say $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_k$ is a chain of length k in $R/(x)$. The inverse images in R give a proper chain of primes in R of length k, all of which contain (x) and thus properly contain 0. Thus adding zero will give a chain of primes in R of length $k+1$.

Conversely, we want to show that if there is a chain of primes in R of length $k+1$, then there is a chain of length k in $R/(x)$ for some $x \in \mathfrak{m} - \{0\}$. Let us write the chain of length $k+1$:

$$\mathfrak{q}_{-1} \subset \mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_k \subset R.$$

Now evidently $\mathfrak{q}_0$ contains some $x \in \mathfrak{m} - 0$. Then the chain $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_k$ can be identified with a chain in $R/(x)$ for this x. So for this x, we have that $\operatorname{Krdim} R \leq \sup \operatorname{Krdim} R/(x) + 1$.▲

There is thus a combinatorial definition of definition.

Geometrically, let $X = \operatorname{Spec} R$ for R an affine ring over $\mathbb{C}$ (a polynomial ring mod some ideal). Then R has Krull dimension $\geq k$ iff there is a chain of irreducible subvarieties of X,

$$X_0 \supset X_1 \supset \cdots \supset X_k.$$

You will meet justification for this in Section 3.6 below.

Remark (Warning!) Let R be a local noetherian ring of dimension k. This means that there is a chain of prime ideals of length k, and no longer chains. Thus there is a maximal chain whose length is k. However, not all maximal chains in $\operatorname{Spec} R$ have length k.

Example 2.5 Let $R = (\mathbb{C}[X,Y,Z]/(XY,XZ))_{(X,Y,Z)}$. It is left as an exercise to the reader to see that there are maximal chains of length not two.

There are more complicated local noetherian *domains* which have maximal chains of prime ideals not of the same length. These examples are not what you would encounter in daily experience, and are necessarily complicated. This cannot happen for finitely generated domains over a field.

Example 2.6 An easier way all maximal chains could fail to be of the same length is if $\operatorname{Spec} R$ has two components (in which case $R = R_0 \times R_1$ for rings R_0, R_1).

2.4 Yet another definition

Let's start by thinking about the definition of a module. Recall that if $(R, \mathfrak{m})$ is a local noetherian ring and M a finitely generated R-module, and $x \in \mathfrak{m}$ is a nonzerodivisor on M, then

$$\dim \operatorname{supp} M/xM = \dim \operatorname{supp} M - 1.$$

Question What if x is a zerodivisor?

This is not necessarily true (e.g. if $x \in \operatorname{Ann}(M)$). Nonetheless, we claim that even in this case:

Proposition 2.7 *For any $x \in \mathfrak{m}$,*

$$\boxed{\dim \operatorname{supp} M \geq \dim \operatorname{supp} M/xM \geq \dim \operatorname{supp} M - 1.}$$

The upper bound on $\dim M/xM$ is obvious as M/xM is a quotient of M. The lower bound is trickier.

Proof. Let $N = \{a \in M : x^n a = 0 \text{ for some } n\}$. We can construct an exact sequence

$$0 \to N \to M \to M/N \to 0.$$

Let $M'' = M/N$. Now x is a nonzerodivisor on M/N by construction. We claim that

$$0 \to N/xN \to M/xM \to M''/xM'' \to 0$$

is exact as well. For this we only need to see exactness at the beginning, i.e. injectivity of $N/xN \to M/xM$. So we need to show that if $a \in N$ and $a \in xM$, then $a \in xN$.

To see this, suppose $a = xb$ where $b \in M$. Then if $\phi : M \to M''$, then $\phi(b) \in M''$ is killed by x as $x\phi(b) = \phi(bx) = \phi(a)$. This means that $\phi(b) = 0$ as $M'' \stackrel{x}{\to} M''$ is injective. Thus $b \in N$ in fact. So $a \in xN$ in fact.

From the exactness, we see that (as x is a nonzerodivisor on M'')

$$\begin{aligned}\dim M/xM &= \max(\dim M''/xM'', \dim N/xN) \geq \max(\dim M'' - 1, \dim N)\\ &\geq \max(\dim M'', \dim N) - 1.\end{aligned}$$

The reason for the last claim is that $\operatorname{supp} N/xN = \operatorname{supp} N$ as N is x-torsion, and the dimension depends only on the support. But the thing on the right is just $\dim M - 1$. ▲

As a result, we find:

Proposition 2.8 $\dim \operatorname{supp} M$ *is the minimal integer* n *such that there exist elements* $x_1, \ldots, x_n \in \mathfrak{m}$ *with* $M/(x_1, \ldots, x_n)M$ *has finite length.*

Note that n always exists, since we can look at a bunch of generators of the maximal ideal, and $M/\mathfrak{m}M$ is a finite-dimensional vector space and is thus of finite length.

Proof. Induction on $\dim \operatorname{supp} M$. Note that $\dim \operatorname{supp}(M) = 0$ if and only if the Hilbert polynomial has degree zero, i.e. M has finite length or that $n = 0$ (n being defined as in the statement).

Suppose $\dim \operatorname{supp} M > 0$.

1. We first show that there are $x_1, \ldots, x_{\dim M}$ with $M/(x_1, \ldots, x_{\dim M})M$ have finite length. Let $M' \subset M$ be the maximal submodule having finite length. There is an exact sequence

   $$0 \to M' \to M \to M'' \to 0$$

   where $M'' = M/M'$ has no finite length submodules. In this case, we can basically ignore M', and replace M by M''. The reason is that modding out by M' doesn't affect either n or the dimension.

   So let us replace M with M'' and thereby assume that M has no finite length submodules. In particular, M does not contain a copy of $R/\mathfrak{m}$, i.e. $\mathfrak{m} \notin \operatorname{Ass}(M)$. By prime avoidance, this means that there is $x_1 \in \mathfrak{m}$ that acts as a nonzerodivisor on M. Thus

   $$\dim M/x_1M = \dim M - 1.$$

   The inductive hypothesis says that there are $x_2, \ldots, x_{\dim M}$ with

   $$(M/x_1M)/(x_2, \ldots, x_{\dim M})(M/xM) \simeq M/(x_1, \ldots, x_{\dim M})M$$

   of finite length. This shows the claim.

2. Conversely, suppose that there $M/(x_1, \ldots, x_n)M$ has finite length. Then we claim that $n \geq \dim M$. This follows because we had the previous result that modding out by a single element can chop off the dimension by at most 1. Recursively applying this, and using the fact that dim of a finite length module is zero, we find

   $$0 = \dim M/(x_1, \ldots, x_n)M \geq \dim M - n.$$ ▲

Corollary 2.9 *Let $(R, \mathfrak{m})$ be a local noetherian ring. Then* $\dim R$ *is equal to the minimal n such that there exist $x_1, \dots, x_n \in R$ with $R/(x_1, \dots, x_n)R$ is artinian. Or, equivalently, such that $(x_1, \dots, x_n)$ contains a power of $\mathfrak{m}$.*

Remark We manifestly have here that the dimension of R is at most the embedding dimension. Here, we're not worried about generating the maximal ideal, but simply something containing a power of it.

We have been talking about dimension. Let R be a local noetherian ring with maximal ideal $\mathfrak{m}$. Then, as we have said in previous lectures, $\dim R$ can be characterized by:

1. The minimal n such that there is an n-primary ideal generated by n elements $x_1, \dots, x_n \in \mathfrak{m}$. That is, the closed point $\mathfrak{m}$ of $\operatorname{Spec} R$ is cut out *set-theoretically* by the intersection $\bigcap V(x_i)$. This is one way of saying that the closed point can be defined by n parameters.

2. The *maximal* n such that there exists a chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n.$$

3. The degree of the Hilbert polynomial $f^+(t)$, which equals $\ell(R/\mathfrak{m}^t)$ for $t \gg 0$.

2.5 Krull's Hauptidealsatz

Let R be a local noetherian ring. The following is now clear from what we have shown:

Theorem 2.10 *R has dimension 1 if and only if there is a nonzerodivisor $x \in \mathfrak{m}$ such that $R/(x)$ is artinian.*

Remark Let R be a domain. We said that a nonzero prime $\mathfrak{p} \subset R$ is **height one** if $\mathfrak{p}$ is minimal among the prime ideals containing some nonzero $x \in R$.

According to Krull's Hauptidealsatz, $\mathfrak{p}$ has height one **if and only if** $\dim R_{\mathfrak{p}} = 1$**.**

We can generalize the notion of $\mathfrak{p}$ as follows.

Definition 2.11 Let R be a noetherian ring (not necessarily local), and $\mathfrak{p} \in \operatorname{Spec} R$. Then we define the **height** of $\mathfrak{p}$, denoted height($\mathfrak{p}$), as $\dim R_{\mathfrak{p}}$. We know that this is the length of a maximal chain of primes in $R_{\mathfrak{p}}$. This is thus the maximal length of prime ideals of R,

$$\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$$

that ends in $\mathfrak{p}$. This is the origin of the term "height."

Remark Sometimes, the height is called the **codimension**. This corresponds to the codimension in $\operatorname{Spec} R$ of the corresponding irreducible closed subset of $\operatorname{Spec} R$.

Theorem 2.12 (Krull's Hauptidealsatz) *Let R be a noetherian ring, and $x \in R$ a nonzerodivisor. If $\mathfrak{p} \in \operatorname{Spec} R$ is minimal over x, then $\mathfrak{p}$ has height one.*

Proof. Immediate from Theorem 2.10. ▲

Theorem 2.13 (Artin-Tate) *Let A be a noetherian domain. Then the following are equivalent:*

1. *There is $f \in A - \{0\}$ such that A_f is a field.*

2. *A has finitely many maximal ideals and has dimension at most 1.*

Proof. We follow [GD].

Suppose first that there is f with A_f a field. Then all nonzero prime ideals of A contain f. We need to deduce that A has dimension ≤ 1. Without loss of generality, we may assume that A is not a field.

There are finitely many primes $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ which are minimal over f; these are all height one. The claim is that any maximal ideal of A is of this form. Suppose $\mathfrak{m}$ were maximal and not one of the $\mathfrak{p}_i$. Then by prime avoidance, there is $g \in \mathfrak{m}$ which lies in no $\mathfrak{p}_i$. A minimal prime $\mathfrak{P}$ of g has height one, so by our assumptions contains f. However, it is then one of the $\mathfrak{p}_i$; this is a contradiction as $g \in \mathfrak{P}$. ▲

2.6 Further remarks

We can recast earlier notions in terms of dimension.

Remark A noetherian ring has dimension zero if and only if R is artinian. Indeed, R has dimension zero iff all primes are maximal.

Remark A noetherian domain has dimension zero iff it is a field. Indeed, in this case (0) is maximal.

Remark R has dimension ≤ 1 if and only if every non-minimal prime of R is maximal. That is, there are no chains of length ≥ 2.

Remark A (noetherian) domain R has dimension ≤ 1 iff every nonzero prime ideal is maximal.

In particular,

Proposition 2.14 *R is Dedekind iff it is a noetherian, integrally closed domain of dimension* 1.

§3 Further topics

3.1 Change of rings

Let $f : R \to R'$ be a map of noetherian rings.

Question What is the relationship between $\dim R$ and $\dim R'$?

A map f gives a map $\operatorname{Spec} R' \to \operatorname{Spec} R$, where $\operatorname{Spec} R'$ is the union of various fibers over the points of $\operatorname{Spec} R$. You might imagine that the dimension is the dimension of R plus the fiber dimension. This is sometimes true.

Now assume that R, R' are *local* with maximal ideals $\mathfrak{m}, \mathfrak{m}'$. Assume furthermore that f is local, i.e. $f(\mathfrak{m}) \subset \mathfrak{m}'$.

Theorem 3.1 $\dim R' \leq \dim R + \dim R'/\mathfrak{m}R'$. *Equality holds if $f : R \to R'$ is flat.*

Here $R'/\mathfrak{m}R'$ is to be interpreted as the "fiber" of $\operatorname{Spec} R'$ above $\mathfrak{m} \in \operatorname{Spec} R$. The fibers can behave weirdly as the basepoint varies in $\operatorname{Spec} R$, so we can't expect equality in general.

Remark Let us review flatness as it has been a while. An R-module M is *flat* iff the operation of tensoring with M is an exact functor. The map $f : R \to R'$ is *flat* iff R' is a flat R-module. Since the construction of taking fibers is a tensor product (i.e. $R'/\mathfrak{m}R' = R' \otimes_R R/\mathfrak{m}$), perhaps the condition of flatness here is not as surprising as it might be.

Proof. Let us first prove the inequality. Say

$$\dim R = a, \ \dim R'/\mathfrak{m}R' = b.$$

We'd like to see that

$$\dim R' \leq a + b.$$

To do this, we need to find $a+b$ elements in the maximal ideal $\mathfrak{m}'$ that generate a $\mathfrak{m}'$-primary ideal of R'.

There are elements $x_1, \dots, x_a \in \mathfrak{m}$ that generate an $\mathfrak{m}$-primary ideal $I = (x_1, \dots, x_a)$ in R. There is a surjection $R'/IR' \twoheadrightarrow R'/\mathfrak{m}R'$. The kernel $\mathfrak{m}R'/IR'$ is nilpotent since I contains a power of $\mathfrak{m}$. We've seen that nilpotents *don't* affect the dimension. In particular,

$$\dim R'/IR' = \dim R'/\mathfrak{m}R' = b.$$

There are thus elements $y_1, \dots, y_b \in \mathfrak{m}'/IR'$ such that the ideal $J = (y_1, \dots, y_b) \subset R'/IR'$ is $\mathfrak{m}'/IR'$-primary. The inverse image of J in R', call it $\overline{J} \subset R'$, is $\mathfrak{m}'$-primary. However, $\overline{J}$ is generated by the $a+b$ elements

$$f(x_1), \dots, f(x_a), \overline{y_1}, \dots, \overline{y_b}$$

if the $\overline{y_i}$ lift y_i.

But we don't always have equality. Nonetheless, if all the fibers are similar, then we should expect that the dimension of the "total space" $\operatorname{Spec} R'$ is the dimension of the "base" $\operatorname{Spec} R$ plus the "fiber" dimension $\operatorname{Spec} R'/\mathfrak{m}R'$. *The precise condition of f flat articulates the condition that the fibers "behave well."* Why this is so is something of a mystery, for now. But for some evidence, take the present result about fiber dimension.

Anyway, let us now prove equality for flat R-algebras. As before, write $a = \dim R, b = \dim R'/\mathfrak{m}R'$. We'd like to show that

$$\dim R' \geq a + b.$$

By what has been shown, this will be enough. This is going to be tricky since we now need to give *lower bounds* on the dimension; finding a sequence $x_1, \dots, x_{a+b}$ such that the quotient $R/(x_1, \dots, x_{a+b})$ is artinian would bound *above* the dimension.

So our strategy will be to find a chain of primes of length $a+b$. Well, first we know that there are primes

$$\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \dots \subset \mathfrak{q}_b \subset R'/\mathfrak{m}R'.$$

Let $\overline{\mathfrak{q}_i}$ be the inverse images in R'. Then the $\overline{\mathfrak{q}_i}$ are a strictly ascending chain of primes in R' where $\overline{\mathfrak{q}_0}$ contains $\mathfrak{m}R'$. So we have a chain of length b; we need to extend this by additional terms.

Now $f^{-1}(\overline{\mathfrak{q}_0})$ contains $\mathfrak{m}$, hence is $\mathfrak{m}$. Since $\dim R = a$, there is a chain $\{\mathfrak{p}_i\}$ of prime ideals of length a going down from $f^{-1}(\overline{\mathfrak{q}_0}) = \mathfrak{m}$. We are now going to find primes $\mathfrak{p}'_i \subset R'$ forming a chain such that $f^{-1}(\mathfrak{p}'_i) = \mathfrak{p}_i$. In other words, we are going to *lift* the chain $\mathfrak{p}_i$ to $\operatorname{Spec} R'$. We can do this at the first stage for $i = a$, where $\mathfrak{p}_a = \mathfrak{m}$ and we can set $\mathfrak{p}'_a = \overline{\mathfrak{q}_0}$. If we can indeed do this lifting, and catenate the chains $\overline{\mathfrak{q}_j}, \mathfrak{p}'_i$, then we will have a chain of the appropriate length.

We will proceed by descending induction. Assume that we have $\mathfrak{p}'_{i+1} \subset R'$ and $f^{-1}(\mathfrak{p}'_{i+1}) = \mathfrak{p}_{i+1} \subset R$. We want to find $\mathfrak{p}'_i \subset \mathfrak{p}'_{i+1}$ such that $f^{-1}(\mathfrak{p}'_i) = \mathfrak{p}_i$. The existence of that prime is a consequence of the following general fact.

Theorem 3.2 (Going down) *Let $f : R \to R'$ be a flat map of noetherian commutative rings. Suppose $\mathfrak{q} \in \operatorname{Spec} R'$, and let $\mathfrak{p} = f^{-1}(\mathfrak{q})$. Suppose $\mathfrak{p}_0 \subset \mathfrak{p}$ is a prime of R. Then there is a prime $\mathfrak{q}_0 \subset \mathfrak{q}$ with*

$$f^{-1}(\mathfrak{q}_0) = \mathfrak{p}_0.$$

Proof. We may replace R' with $R'_{\mathfrak{q}}$. There is still a map

$$R \to R'_{\mathfrak{q}}$$

which is flat as localization is flat. The maximal ideal in $R'_{\mathfrak{q}}$ has inverse image $\mathfrak{p}$. So the problem now reduces to finding *some* $\mathfrak{p}_0$ in the localization that pulls back appropriately.

Anyhow, throwing out the old R and replacing with the localization, we may assume that R' is local and $\mathfrak{q}$ the maximal ideal. (The condition $\mathfrak{q}_0 \subset \mathfrak{q}$ is now automatic.)

The claim now is that we can replace R with $R/\mathfrak{p}_0$ and R' with $R'/\mathfrak{p}_0 R' = R' \otimes R/\mathfrak{p}_0$. We can do this because base change preserves flatness (see below), and in this case we can reduce to the case of $\mathfrak{p}_0 = (0)$—in particular, R is a domain. Taking these quotients just replaces $\mathrm{Spec}\, R, \mathrm{Spec}\, R'$ with closed subsets where all the action happens anyhow.

Under these replacements, we now have:

1. R' is local with maximal ideal $\mathfrak{q}$
2. R is a domain and $\mathfrak{p}_0 = (0)$.

We want a prime of R' that pulls back to (0) in R. I claim that any minimal prime of R' will work. Suppose otherwise. Let $\mathfrak{q}_0 \subset R'$ be a minimal prime, and suppose $x \in R \cap f^{-1}(\mathfrak{q}_0) - \{0\}$. But $\mathfrak{q}_0 \in \mathrm{Ass}(R')$. So $f(x)$ is a zerodivisor on R'. Thus multiplication by x on R' is not injective.

But, R is a domain, so $R \stackrel{x}{\to} R$ is injective. Tensoring with R' must preserve this, implying that $R' \stackrel{x}{\to} R'$ is injective because R' is flat. This is a contradiction. ▲

We used:

Lemma 3.3 *Let $R \to R'$ be a flat map, and S an R-algebra. Then $S \to S \otimes_R R'$ is a flat map.*

Proof. The construction of taking an S-module with $S \otimes_R R'$ is an exact functor, because that's the same thing as taking an S-module, restricting to R, and tensoring with R'. ▲

The proof of the fiber dimension theorem is now complete.

3.2 The dimension of a polynomial ring

Adding an indeterminate variable corresponds geometrically to taking the product with the affine line, and so should increase the dimension by one. We show that this is indeed the case.

Theorem 3.4 *Let R be a noetherian ring. Then* $\dim R[X] = \dim R + 1$.

Interestingly, this is *false* if R is non-noetherian, cf. []. Let R be a ring of dimension n.

Lemma 3.5 $\dim R[x] \geq \dim R + 1$.

Proof. Let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ be a chain of primes of length $n = \dim R$. Then $\mathfrak{p}_0 R[x] \subset \cdots \subset \mathfrak{p}_n R[x] \subset (x, \mathfrak{p}_n) R[x]$ is a chain of primes in $R[x]$ of length $n+1$ because of the following fact: if $\mathfrak{q} \subset R$ is prime, then so is $\mathfrak{q} R[x] \subset R[x]$.[2] Note also that as $\mathfrak{p}_n \subsetneq R$, we have that $\mathfrak{p}_n R[x] \subsetneq (x, \mathfrak{p}_n)$. So this is indeed a legitimate chain. ▲

Now we need only show:

Lemma 3.6 *Let R be noetherian of dimension n. Then* $\dim R[x] \leq \dim R + 1$.

[2]This is because $R[x]/\mathfrak{q}R[x] = (R/\mathfrak{q})[x]$ is a domain.

Proof. Let $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_m \subset R[x]$ be a chain of primes in $R[x]$. Let $\mathfrak{m} = \mathfrak{q}_m \cap R$. Then if we localize and replace R with $R_{\mathfrak{m}}$, we get a chain of primes of length m in $R_{\mathfrak{m}}[x]$. In fact, we get more. We get a chain of primes of length m in $(R[x])_{\mathfrak{q}_m}$, and a *local* inclusion of noetherian local rings

$$R_{\mathfrak{m}} \hookrightarrow (R[x])_{\mathfrak{q}_m}.$$

To this we can apply the fiber dimension theorem. In particular, this implies that

$$m \leq \dim(R[x])_{\mathfrak{q}_m} \leq \dim R_{\mathfrak{m}} + \dim(R[x])_{\mathfrak{q}_m}/\mathfrak{m}(R[x])_{\mathfrak{q}_m}.$$

Here $\dim R_{\mathfrak{m}} \leq \dim R = n$. So if we show that $\dim(R[x])_{\mathfrak{q}_m}/\mathfrak{m}(R[x])_{\mathfrak{q}_m} \leq 1$, we will have seen that $m \leq n+1$, and will be done. But this last ring is a localization of $(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}})[x]$, which is a PID by the euclidean algorithm for polynomial rings over a field, and thus of dimension ≤ 1. ▲

3.3 A refined fiber dimension theorem

Let R be a local noetherian domain, and let $R \to S$ be an injection of rings making S into an R-algebra. Suppose S is also a local domain, such that the morphism $R \to S$ is local. This is essentially the setup of Section 3.2, but in this section, we make the refining assumption that S is *essentially of finite type* over R; in other words, S is the localization of a finitely generated R-algebra.

Let k be the residue field of R, and k' that of S; because $R \to S$ is local, there is induced a morphism of fields $k \to k'$. We shall prove, following [GD]:

Theorem 3.7 (Dimension formula)

$$\dim S + \text{tr.deg.} S/R \leq \dim R + \text{tr.deg.} k'/k. \tag{10.1}$$

Here tr.deg.B/A is more properly the transcendence degree of the quotient field of B over that of A. Geometrically, it corresponds to the dimension of the "generic" fiber.

Proof. Let $\mathfrak{m} \subset R$ be the maximal ideal. We know that S is a localization of an algebra of the form $(R[x_1, \dots, x_k])/\mathfrak{p}$ where $\mathfrak{p} \subset R[x_1, \dots, x_n]$ is a prime ideal $\mathfrak{q}$. We induct on k.

Since we can "dévissage" the extension $R \to S$ as the composite

$$R \to (R[x_1, \dots, x_{k-1}]/(\mathfrak{p} \cap R[x_1, \dots, x_{k-1}])_{\mathfrak{q}'} \to S,$$

(where $\mathfrak{q}' \in \operatorname{Spec} R[x_1, \dots, x_{k-1}]/(\mathfrak{p} \cap R[x_1, \dots, x_{k-1}]$ is the pull-back of $\mathfrak{q}$), we see that it suffices to prove (10.1) when $k = 1$, that is S is the localization of a quotient of $R[x]$.

So suppose $k = 1$. Then we have $S = (R[x])_{\mathfrak{q}}/\mathfrak{p}$ where $\mathfrak{q} \subset R[x]$ is another prime ideal lying over $\mathfrak{m}$. Let us start by considering the case where $\mathfrak{q} = 0$.

Lemma 3.8 *Let $(R, \mathfrak{m})$ be a local noetherian domain as above. Let $S = R[x]_{\mathfrak{q}}$ where $\mathfrak{q} \in \operatorname{Spec} R[x]$ is a prime lying over $\mathfrak{m}$. Then* (10.1) *holds with equality.*

Proof. In this case, tr.deg.$S/R = 1$. Now $\mathfrak{q}$ could be $\mathfrak{m}R[x]$ or a prime ideal containing that, which is then automatically maximal, as we know from the proof of Section 3.2. Indeed, primes containing $\mathfrak{m}R[x]$ are in bijection with primes of $R/\mathfrak{m}[x]$, and these come in two forms: zero, and those generated by one element. (Note that in the former case, the residue field is the field of rational functions $k(x)$ and in the second, the residue field is finite over k.)

1. In the first case, $\dim S = \dim R[x]_{\mathfrak{m}R[x]} = \dim R$ and but the residue field extension is $(R[x]_{\mathfrak{m}R[x]})/\mathfrak{m}R[x]_{\mathfrak{m}R[x]} = k(x)$, so tr.deg.$k'/k = 1$ and the formula is satisfied.

2. In the second case, $\mathfrak{q}$ properly contains $\mathfrak{m}R[x]$. Then $\dim R[x]_{\mathfrak{q}} = \dim R + 1$, but the residue field extension is finite. So in this case too, the formula is satisfied. ▲

Now, finally, we have to consider the case where $\mathfrak{p} \subset R[x]$ is not zero, and we have $S = (R[x]/\mathfrak{p})_{\mathfrak{q}}$ for $\mathfrak{q} \in \operatorname{Spec} R[x]/\mathfrak{p}$ lying over $\mathfrak{m}$. In this case, tr.deg.$S/R = 0$. So we need to prove

$$\dim S \leq \dim R + \text{tr.deg.} k'/k.$$

Let us, by abuse of notation, identify $\mathfrak{q}$ with its preimage in $R[x]$. (Recall that $\operatorname{Spec} R[x]/\mathfrak{p}$ is canonically identified as a closed subset of $\operatorname{Spec} R[x]$.) Then we know that $\dim(R[x]/\mathfrak{p})_{\mathfrak{q}}$ is the largest chain of primes in $R[x]$ between $\mathfrak{p}, \mathfrak{q}$. In particular, it is at most $\dim R[x]_{\mathfrak{q}} - \text{height}\mathfrak{p} \leq \dim R + 1 - 1 = \dim R$. So the result is clear. ▲

In [GD], this is used to prove the geometric result that if $\phi : X \to Y$ is a morphism of varieties over an algebraically closed field (or a morphism of finite type between nice schemes), then the local dimension (that is, the dimension at x) of the fiber $\phi^{-1}(\phi(x))$ is an upper semi-continuous function of $x \in X$.

3.4 An infinite-dimensional noetherian ring

We shall now present an example, due to Nagata, of an infinite-dimensional noetherian ring. Note that such a ring cannot be *local*.

Consider the ring $R = \mathbb{C}[\{x_{i,j}\}_{0 \leq i \leq j}]$ of polynomials in infinitely many variables $x_{i,j}$. This is clearly an infinite-dimensional ring, but it is also not noetherian. We will localize it suitably to make it noetherian.

Let $\mathfrak{p}_n \subset R$ be the ideal $(x_{1,n}, x_{2,n}, \ldots, x_{n,n})$ for all $i \leq n$. Let $S = R - \bigcup \mathfrak{p}_n$; this is a multiplicatively closed set.

Theorem 3.9 (Nagata) *The ring $S^{-1}R$ is noetherian and has infinite dimension.*

We start with

Proposition 3.10 *The ring in the statement of the problem is noetherian.*

The proof is slightly messy, so we first prove a few lemmas.

Let $R' = S^{-1}R$ as in the problem statement. We start by proving that every ideal in R' is contained in one of the $\mathfrak{p}_n$ (which, by abuse of notation, we identify with their localizations in $R' = S^{-1}R$). In particular, the $\mathfrak{p}_n$ are the maximal ideals in R'.

Lemma 3.11 *The $\mathfrak{p}_n$ are the maximal ideals in R'.*

Proof. We start with an observation:

> If $f \neq 0$, then f belongs to only finitely many $\mathfrak{p}_n$.

To see this, let us use the following notation. If M is a monomial, we let $S(M)$ denote the set of subscripts $x_{a,b}$ that occur and $S_2(M)$ the set of second subscripts (i.e. the b's). For $f \in R$, we define $S(f)$ to be the intersection of the $S(M)$ for M a monomial occurring nontrivially in f. Similarly we define $S_2(f)$.

Let us prove:

Lemma 3.12 *$f \in \mathfrak{p}_n$ iff $n \in S_2(f)$. Moreover, $S(f)$ and $S_2(f)$ are finite for any $f \neq 0$.*

Proof. Indeed, $f \in \mathfrak{p}_n$ iff every monomial in f is divisible by some $x_{i,n}, i \leq n$, as $\mathfrak{p}_n = (x_{i,n})_{i \leq n}$. From this the first assertion is clear. The second too, because f will contain a nonzero monomial, and that can be divisible by only finitely many $x_{a,b}$. ▲

From this, it is clear how to define $S_2(f)$ for any element in R', not necessarily a polynomial in R. Namely, it is the set of n such that $f \in \mathfrak{p}_n$. It is now clear, from the second statement of the lemma, that any $f \neq 0$ lies in *only finitely many* $\mathfrak{p}_n$. In particular, the observation has been proved.

Let $\mathcal{T} = \{S_2(f), f \in I - 0\}$. *I claim that* $\emptyset \in \mathcal{T}$ *iff* $I = (1)$. For $\emptyset \in \mathcal{T}$ iff there is a polynomial lying in no $\mathfrak{p}_n$. Since the union $\bigcup \mathfrak{p}_n$ is the set of non-units (by construction), we find that the assertion is clear.

Lemma 3.13 *$\mathcal{T}$ is closed under finite intersections.*

Proof. Suppose $T_1, T_2 \in \mathcal{T}$. Without loss of generality, there are *polynomials* $F_1, F_2 \in R$ such that $S_2(F_1) = T_1, S_2(F_2) = T_2$. A generic linear combination $aF_1 + bF_2$ will involve no cancellation for $a, b \in \mathbb{C}$, and the monomials in this linear combination will be the union of those in F_1 and those in F_2 (scaled appropriately). So $S_2(aF_1 + bF_2) = S_2(F_1) \cap S_2(F_2)$. ▲

Finally, we can prove the result that the $\mathfrak{p}_n$ are the only maximal ideals. Suppose I was contained in no $\mathfrak{p}_n$, and form the set $\mathcal{T}$ as above. This is a collection of finite sets. Since $I \not\subset \mathfrak{p}_n$ for each n, we find that $n \notin \bigcap_{T \in \mathcal{T}} T$. This intersection is thus empty. It follows that there is a *finite* intersection of sets in $\mathcal{T}$ which is empty as $\mathcal{T}$ consists of finite sets. But $\mathcal{T}$ is closed under intersections. There is thus an element in I whose S_2 is empty, and is thus a unit. Thus $I = (1)$.▲

We have proved that the $\mathfrak{p}_n$ are the only maximal ideals. This is not enough, though. We need:

Lemma 3.14 *$R'_{\mathfrak{p}_n}$ is noetherian for each n.*

Proof. Indeed, any polynomial involving the variables $x_{a,b}$ for $b \neq n$ is invertible in this ring. We see that this ring contains the field

$$\mathbb{C}(\{x_{a,b}, b \neq n\}),$$

and over it is contained in the field $\mathbb{C}(\{x_{a,b}, \forall a, b\})$. It is a localization of the algebra $\mathbb{C}(\{x_{a,b}, b \neq n\})[x_{1,n}, \dots, x_{n,n}]$ and is consequently noetherian by Hilbert's basis theorem. ▲

The proof will be completed with:

Lemma 3.15 *Let R be a ring. Suppose every element $x \neq 0$ in the ring belongs to only finitely many maximal ideals, and suppose that $R_{\mathfrak{m}}$ is noetherian for each $\mathfrak{m} \subset R$ maximal. Then R is noetherian.*

Proof. Let $I \subset R$ be a nonzero ideal. We must show that it is finitely generated. We know that I is contained in only finitely many maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_k$. At each of these maximal ideals, we know that $I_{\mathfrak{m}_i}$ is finitely generated. Clearing denominators, we can choose a finite set of generators in R. So we can collect them together and get a finite set $a_1, \dots, a_N \subset I$ which generate $I_{\mathfrak{m}_i} \subset R_{\mathfrak{m}_i}$ for each i. It is not necessarily true that $J = (a_1, \dots, a_N) = I$, though we do have $\subset$. However, $I_{\mathfrak{m}} = J_{\mathfrak{m}}$ except at finitely many maximal ideals $\mathfrak{n}_1, \dots, \mathfrak{n}_M$ because a nonzero element is a.e. a unit. However, these $\mathfrak{n}_j$ are not among the $\mathfrak{m}_i$. In particular, for each j, there is $b_j \in I - \mathfrak{n}_j$ as $I \not\subset \mathfrak{n}_j$. Then we find that the ideal

$$(a_1, \dots, a_N, b_1, \dots, b_M) \subset I$$ ▲

becomes equal to I in all the localizations. So it is I, and I is finitely generated

We need only see that the ring R' has infinite dimension. But for each n, there is a chain of primes $(x_{1,n}) \subset (x_{1,n}, x_{2,n}) \subset \cdots \subset (x_{1,n}, \dots, x_{n,n})$ of length $n - 1$. The supremum of the lengths is thus infinite.

3.5 Catenary rings

Definition 3.16 A ring R is *catenary* if given any two primes $\mathfrak{p} \subsetneq \mathfrak{p}'$, any two maximal prime chains from $\mathfrak{p}$ to $\mathfrak{p}'$ have the same length.

Nagata showed that there are noetherian domains which are not catenary. We shall see that *affine rings*, or rings finitely generated over a field, are always catenary.

Definition 3.17 If $\mathfrak{p} \in \operatorname{Spec} R$, then $\dim \mathfrak{p} := \dim R/\mathfrak{p}$.

Lemma 3.18 *Let S be a k-affine domain with $tr.d._k S = n$, and let $\mathfrak{p} \in \operatorname{Spec} S$ be height one. Then $tr.d._k(S/\mathfrak{p}) = n - 1$.*

Proof. Case 1: assume $S = k[x_1, \ldots, x_n]$ is a polynomial algebra. In this case, height 1 primes are principal, so $\mathfrak{p} = (f)$ for some f. Say f has positive degree with respect to x_1, so $f = g_r(x_2, \ldots, x_n)x_1^r + \cdots$. We have that $k[x_2, \ldots, x_n] \cap (f) = (0)$ (just look at degree with respect to x_1). It follows that $k[x_2, \ldots, x_n] \hookrightarrow S/(f)$, so $\bar{x}_2, \ldots, \bar{x}_n$ are algebraically independent in $S/\mathfrak{p}$. By $\bar{x}_1$ is algebraic over $Q(k[\bar{x}_2, \ldots, \bar{x}_n])$ as witnessed by f. This, $tr.d._k S/\mathfrak{p} = n - 1$.

Case 2: reduction to case 1. Let $R = k[x_1, \ldots, x_n]$ be a Noether normalization for S, and let $\mathfrak{p}_0 = \mathfrak{p} \cap R$. Observe that Going Down applies (because S is a domain and R is normal). It follows that $ht_R(\mathfrak{p}_0) = ht_S(\mathfrak{p}) = 1$. By case 1, we get that $tr.d.(R/\mathfrak{p}_0) = n - 1$. By $(*)$, we get that $tr.d.R/\mathfrak{p}_0 = tr.d.(S/\mathfrak{p})$. ▲

Theorem 3.19 *Any k-affine algebra S is catenary (even if S is not a domain). In fact, any saturated prime chain from $\mathfrak{p}$ to $\mathfrak{p}'$ has length $\dim \mathfrak{p} - \dim \mathfrak{p}'$. If S is a domain, then all maximal ideals have the same height.*

Proof. Consider any chain $\mathfrak{p} \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}'$. Then we get the chain

$$S/\mathfrak{p} \twoheadrightarrow S/\mathfrak{p}_1 \twoheadrightarrow \cdots \twoheadrightarrow S/\mathfrak{p}_r = S/\mathfrak{p}'$$

Here $\mathfrak{p}_i/\mathfrak{p}_{i-1}$ is height 1 in $S/\mathfrak{p}_{i-1}$, so each arrow decreases the transcendence degree by exactly 1. Therefore, $tr.d._k S/\mathfrak{p}' = tr.d._k S/\mathfrak{p} - r$.

$$r = tr.d._k S/\mathfrak{p} - tr.d._k S/\mathfrak{p}' = \dim S/\mathfrak{p} - \dim S/\mathfrak{p}' = \dim \mathfrak{p} - \dim \mathfrak{p}'.$$

To get the last statement, take $\mathfrak{p} = 0$ and $\mathfrak{p}' = \mathfrak{m}$. Then we get that $ht(\mathfrak{m}) = \dim S$. ▲

Note that the last statement fails in general.

Example 3.20 Take $S = k \times k[x_1, \ldots, x_n]$. Then $ht(0 \times k[x_1, \ldots, x_n]) = 0$, but $ht\big(k \times (x_1, \ldots, x_n)\big) = n$.

But that example is not connected.

Example 3.21 $S = k[x, y, z]/(xy, xz)$.

But this example is not a domain. In general, for any prime $\mathfrak{p}$ in any ring S, we have

$$ht(\mathfrak{p}) + \dim \mathfrak{p} \leq \dim S.$$

Theorem 3.22 *Let S be an affine algebra, with minimal primes $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Then the following are equivalent.*

1. $\dim \mathfrak{p}_i$ *are all equal.*
2. $ht(\mathfrak{p}) + \dim \mathfrak{p} = \dim S$ *for all primes* $\mathfrak{p} \in \operatorname{Spec} S$. *In particular, if* S *is a domain, we get this condition.*

Proof. ($1 \Rightarrow 2$) $ht(\mathfrak{p})$ is the length of some saturated prime chain from $\mathfrak{p}$ to some minimal prime $\mathfrak{p}_i$. This length is $\dim \mathfrak{p}_i - \dim \mathfrak{p} = \dim S - \dim \mathfrak{p}$ (by condition 1). Thus, we get (2).

($2 \Rightarrow 1$) Apply (2) to the minimal prime $\mathfrak{p}_i$ to get $\dim \mathfrak{p}_i = \dim S$ for all i. ▲

We finish with a (non-affine) noetherian domain S with maximal ideals of different heights. We need the following fact.
Fact: If R is a ring with $a \in R$, then there is a canonical R-algebra isomorphism $R[x]/(ax-1) \cong R[a^{-1}]$, $x \leftrightarrow a^{-1}$.

Example 3.23 Let $\big(R, (\mathfrak{p}i)\big)$ be a DVR with quotient field K. Let $S = R[x]$, and assume for now that we know that $\dim S = 2$. Look at $\mathfrak{m}_2 = (\mathfrak{p}i, x)$ and $\mathfrak{m}_1 = (\mathfrak{p}ix - 1)$. Note that $\mathfrak{m}_1$ is maximal because $S/\mathfrak{m}_1 = K$. It is easy to show that $ht(\mathfrak{m}_1) = 1$. However, $\mathfrak{m}_2 \supsetneq (x) \supsetneq (0)$, so $ht(\mathfrak{m}_2) = 2$.

3.6 Dimension theory for topological spaces

The present subsection (which consists mostly of exercises) is a digression that may illuminate the notion of Krull dimension.

Definition 3.24 Let X be a topological space.[3] Recall that X is **irreducible** if cannot be written as the union of two proper closed subsets $F_1, F_2 \subsetneq X$.

We say that a subset of X is irreducible if it is irreducible with respect to the induced topology.

In general, this notion is not valid from the topological spaces familiar from analysis. For instance:

Exercise 10.1 Points are the only irreducible subsets of $\mathbb{R}$.

Nonetheless, irreducible sets behave very nicely with respect to certain operations. As you will now prove, if $U \subset X$ is an open subset, then the irreducible closed subsets of U are in bijection with the irreducible closed subsets of X that intersect U.

Exercise 10.2 A space is irreducible if and only if every open set is dense, or alternatively if every open set is connected.

Exercise 10.3 Let X be a space, $Y \subset X$ an irreducible subset. Then $\overline{Y} \subset X$ is irreducible.

Exercise 10.4 Let X be a space, $U \subset X$ an open subset. Then the map $Z \to Z \cap U$ gives a bijection between the irreducible closed subsets of X meeting U and the irreducible closed subsets of U. The inverse is given by $Z' \to \overline{Z'}$.

As stated above, the notion of irreducibility is useless for spaces like manifolds. In fact, by **??** 10.2, a Hausdorff space cannot be irreducible unless it consists of one point. However, for the highly non-Hausdorff spaces encountered in algebraic geometry, this notion is very useful.

Let R be a commutative ring, and $X = \operatorname{Spec} R$.

Exercise 10.5 A closed subset $F \subset \operatorname{Spec} R$ is irreducible if and only if it can be written in the form $F = V(\mathfrak{p})$ for $\mathfrak{p} \subset R$ prime. In particular, $\operatorname{Spec} R$ is irreducible if and only if R has one minimal prime.

[3] We do not include the empty space.

In fact, spectra of rings are particularly nice: they are **sober spaces.**

Definition 3.25 A space X is called **sober** if to every irreducible closed $F \subset X$, there is a unique point $\xi \in F$ such that $F = \overline{\{\xi\}}$. This point is called the **generic point.**

Exercise 10.6 Check that if X is any topological space and $\xi \in X$, then the closure $\overline{\{\xi\}}$ of the point ξ is irreducible.

Exercise 10.7 Show that $\operatorname{Spec} R$ for R a ring is sober.

Exercise 10.8 Let X be a space with a cover $\{X_\alpha\}$ by open subsets, each of which is a sober space. Then X is a sober space. (Hint: any irreducible closed subset must intersect one of the X_α, so is the closure of its intersection with that one.)

We now come to the main motivation of this subsection, and the reason for its inclusion here.

Definition 3.26 Let X be a topological space. Then the **dimension** (or **combinatorial dimension**) of X is the maximal k such that a chain

$$F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_k \subset X$$

with the F_i irreducible, exists. This number is denoted $\dim X$ and may be infinite.

Exercise 10.9 What is the Krull dimension of $\mathbb{R}$?

Exercise 10.10 Let $X = \bigcup X_i$ be the finite union of subsets $X_i \subset X$.

Exercise 10.11 Let R be a ring. Then $\dim \operatorname{Spec} R$ is equal to the Krull dimension of R.

Most of the spaces one wishes to work with in standard algebraic geometry have a strong form of compactness. Actually, compactness is the wrong word, since the spaces of algebraic geometry are not Hausdorff.

Definition 3.27 A space is **noetherian** if every descending sequence of closed subsets $F_0 \supset F_1 \supset \dots$ stabilizes.

Exercise 10.12 If R is noetherian, $\operatorname{Spec} R$ is noetherian as a topological space.

3.7 The dimension of a tensor product of fields

The following very clear result gives us the dimension of the tensor product of fields.

Theorem 3.28 (Grothendieck-Sharp) *Let K, L be field extensions of a field k. Then*

$$\dim K \otimes_k L = \min(\text{tr.deg.} K, \text{tr.deg.} L).$$

This result is stated in the errata of [GD], vol IV (4.2.1.5), but that did not make it well-known; apparently it was independently discovered and published again by R. Y. Sharp, ten years later.[4] Note that in general, this tensor product is *not* noetherian.

[4] Thanks to Georges Elencwajg for a helpful discussion at `http://math.stackexchange.com/questions/56669/a-tensor-product-of-a-power-series/56794`.

Proof. We start by assuming K is a finitely generated, purely transcendental extension of k. Then K is the quotient field of a polynomial ring $k[x_1, \dots, x_n]$. It follows that $K \otimes_k L$ is a localization of $L[x_1, \dots, x_n]$, and consequently of dimension at most $n = \mathrm{tr.deg}.K$.

Now the claim is that if $\mathrm{tr.deg}.L > n$, then we have equality

$$\dim K \otimes_k L = n.$$

To see this, we have to show that $K \otimes_k L$ admits an L-homomorphism to L. For then there will be a maximal ideal $\mathfrak{m}$ of $K \otimes_k L$ which comes from a maximal ideal $\mathfrak{M}$ of $L[x_1, \dots, x_n]$ (corresponding to this homomorphism). Consequently, we will have $(K \otimes_k L)_{\mathfrak{m}} = (L[x_1, \dots, x_n])_{\mathfrak{M}}$, which has dimension n.

So we need to produce this homomorphism $K \otimes_k L \to L$. Since $K = k(x_1, \dots, x_n)$ and L has transcendence degree more than n, we just choose n algebraically independent elements of L, and use that to define a map of k-algebras $K \to L$. By the universal property of the tensor product, we get a section $K \otimes_k L \to L$. This proves the result in the case where K is a finitely generated, purely transcendental extension.

Now we assume that K has finite transcendence degree over k, but is not necessarily purely transcendental. Then K contains a subfield E which is purely transcendental over k and such that E/K is algebraic. Then $K \otimes_k L$ is *integral* over its subring $E \otimes_k L$. The previous analysis applies to $E \otimes_k L$, and by integrality the dimensions of the two objects are the same.

Finally, we need to consider the case when K is allowed to have infinite transcendence degree over k. Again, we may assume that K is the quotient field of the polynomial ring $k[\{x_\alpha\}]$ (by the integrality argument above). We need to show that if L has *larger* transcendence degree than K, then $\dim K \otimes_k L = \infty$. As before, there is a section $K \otimes_k L \to L$, and $K \otimes_k L$ is a localization of the polynomial ring $L[\{x_\alpha\}]$. If we take the maximal ideal in $L[\{x_\alpha\}]$ corresponding to this section $K \otimes_k L \to L$, it is of the form $(x_\alpha - t_\alpha)_\alpha$ for the $t_\alpha \in L$. It is easy to check that the localization of $L[\{x_\alpha\}]$ at this maximal ideal, which is a localization of $K \otimes_k L$, has infinite dimension. ▲

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