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Chapter 18
Complete local rings

This chapter (barely started) is intended to give various results about complete local rings (e.g. the Cohen structure theorems) and results relating rings to their completions (e.g. material on excellent rings).

§1 The Cohen structure theorem

We want now a classification of complete local rings containing a field; it states that they are the homomorphic images of power series rings:

Theorem 1.1 (Cohen structure theorem) *Let $(R, \mathfrak{m})$ be a complete local noetherian ring, which contains a field. Then $R \simeq K[[x_1, \dots, x_n]]/I$ for some ideal $I \subset K[[x_1, \dots, x_n]]$ and some field K.*

We have already shown that this result is true when R contains a copy of its own residue field; that is, when the map $R \to R/\mathfrak{m}$ admits a section.

We are going to show that this is the case if R contains a field.

Remark The condition that R should contain a field means that if $R/\mathfrak{m}$ is of characteristic p, then $pR = 0$. For, if $p > 0$, then R contains a copy of $\mathbb{Z}/p\mathbb{Z}$. If $p = 0$, R automatically contains a copy of $\mathbb{Q}$, as each $n \in \mathbb{Z} - \{0\}$ is automatically invertible in R.

Proof. We just need to show that R contains a copy of its own residue field. To do this, let κ be the prime field contained in R, so $\kappa = \mathbb{Q}$ or $\mathbb{F}_p$. Let k be the residue field $R/\mathfrak{m}$. We have a diagram:

$$\begin{array}{ccc} & \nearrow & R \\ & & \downarrow \\ R/\mathfrak{m} & \xrightarrow{\text{id}} & R/\mathfrak{m}, \end{array}$$

in which we seek a lift; then we can apply **??**, because R will contain a copy of its residue field.

Now, R is complete, so $R = \varprojlim R/\mathfrak{m}^i$. So it suffices to give a compatible sequence of lifts

$$\begin{array}{ccc} & \nearrow & R/\mathfrak{m}^i \ , \\ & & \downarrow \\ R/\mathfrak{m} & \xrightarrow{\text{id}} & R/\mathfrak{m}, \end{array}$$

which we will show exists. That is, given a section $R/\mathfrak{m} \to R/\mathfrak{m}^i$, we will lift it to a section $R/\mathfrak{m} \to R/\mathfrak{m}^{i+1}$, and these will glue to give the desired section $R/\mathfrak{m} \to R$.

Now, everything here is a κ-algebra, and we are looking at a nilpotent lifting property for κ-algebras. It follows that if we can prove that $R/\mathfrak{m}$ is *formally smooth* over κ, then we will be able to do the lifting at each stage, and R will contain a copy of its residue field. Thus, we must show:

Proposition 1.2 *Let κ be a perfect field, and let K/κ be any field extension. Then K is formally smooth over κ.*

Proof. To see this, we will use the fact that K/κ is *separably generated.* Namely, there is a transcendence basis $T \subset K$ such that $K/\kappa(\{T\})$ is a separable *algebraic* extension. We will show that $K/\kappa(\{T\})$ and $\kappa(\{T\})/\kappa$ are each formally smooth, which will imply the result.

Now, we know that $\kappa(\{T\})/\kappa$ is formally smooth: it is the localization of a formally smooth κ-algebra (the polynomial algebra $\kappa[\{T\}]$), and localization is always formally smooth.

Similarly, $K/\kappa(\{T\})$ is formally smooth because any separable algebraic extension is in fact formally étale. We have shown this for finite algebraic extensions (**??**), and a limiting argument establishes it for infinite algebraic extensions. Namely, if A is a $\kappa(\{T\})$-algebra and I a square-zero ideal, then if we have a map $K \to A/I$, the restriction to each finite subextension lifts *uniquely* to A. As a result of this uniqueness, we can glue the liftings to get a lift of $K \to A/I$ to $K \to A$. ▲

Corollary 1.3 *Let $(R, \mathfrak{m})$ be a local ring containing a copy of its residue field k. Then R is formally smooth over k if and only if R is geometrically regular: that is, all the localizations of $R \otimes_k \overline{k}$ are regular.*

Proof. If R is formally smooth over k, then $R \otimes_k \overline{k}$ is formally smooth over $\overline{k}$ and consequently all the localizations are regular. ▲

Ok, need to think some more about this...I am currently leaving it as a comment.

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