Contents

Chapter 0
Categories

The language of categories is not strictly necessary to understand the basics of commutative algebra. Nonetheless, it is extremely convenient and powerful. It will clarify many of the constructions made in the future when we can freely use terms like “universal property” or “adjoint functor.” As a result, we begin this book with a brief introduction to category theory. We only scratch the surface; the interested reader can pursue further study in [ML98] or [KS06].

Nonetheless, the reader is advised not to take the present chapter too seriously; skipping it for the moment to chapter 1 and returning here as a reference could be quite reasonable.

§1 Introduction

1.1 Definitions

Categories are supposed to be places where mathematical objects live. Intuitively, to any given type of structure (e.g. groups, rings, etc.), there should be a category of objects with that structure. These are not, of course, the only type of categories, but they will be the primary ones of concern to us in this book.

The basic idea of a category is that there should be objects, and that one should be able to map between objects. These mappings could be functions, and they often are, but they don't have to be. Next, one has to be able to compose mappings, and associativity and unit conditions are required. Nothing else is required.

Definition 1.1 A **category** $\mathcal{C}$ consists of:

1. A collection of **objects**, $\mathrm{ob}\,\mathcal{C}$.
2. For each pair of objects $X, Y \in \mathrm{ob}\,\mathcal{C}$, a set of **morphisms** $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (abbreviated $\mathrm{Hom}(X, Y)$).
3. For each object $X \in \mathrm{ob}\,\mathcal{C}$, there is an **identity morphism** $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ (often just abbreviated to 1).
4. There is a **composition law** $\circ : \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z), (g, f) \to g \circ f$ for every triple X, Y, Z of objects.
5. The composition law is unital and associative. In other words, if $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, then $1_Y \circ f = f \circ 1_X = f$. Moreover, if $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$ for objects Z, Y, W, then
$$h \circ (g \circ f) = (h \circ g) \circ f \in \mathrm{Hom}_{\mathcal{C}}(X, W).$$

We shall write $f : X \to Y$ to denote an element of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. In practice, $\mathcal{C}$ will often be the storehouse for mathematical objects: groups, Lie algebras, rings, etc., in which case these "morphisms" will just be ordinary functions.

Here is a simple list of examples.

Example 1.2 (Categories of structured sets)

1. $\mathcal{C} = \mathbf{Sets}$; the objects are sets, and the morphisms are functions.
2. $\mathcal{C} = \mathbf{Grps}$; the objects are groups, and the morphisms are maps of groups (i.e. homomorphisms).
3. $\mathcal{C} = \mathbf{LieAlg}$; the objects are Lie algebras, and the morphisms are maps of Lie algebras (i.e. homomorphisms).[1]
4. $\mathcal{C} = \mathbf{Vect}_k$; the objects are vector spaces over a field k, and the morphisms are linear maps.
5. $\mathcal{C} = \mathbf{Top}$; the objects are topological spaces, and the morphisms are continuous maps.
6. This example is slightly more subtle. Here the category $\mathcal{C}$ has objects consisting of topological spaces, but the morphisms between two topological spaces X, Y are the *homotopy classes* of maps $X \to Y$. Since composition respects homotopy classes, this is well-defined.

In general, the objects of a category do not have to form a set; they can be too large for that. For instance, the collection of objects in **Sets** does not form a set.

Definition 1.3 A category is **small** if the collection of objects is a set.

The standard examples of categories are the ones above: structured sets together with structure-preserving maps. Nonetheless, one can easily give other examples that are not of this form.

Example 1.4 (Groups as categories) Let G be a finite group. Then we can make a category B_G where the objects just consist of one element $*$ and the maps $* \to *$ are the elements $g \in G$. The identity is the identity of G and composition is multiplication in the group.

In this case, the category does not represent much of a class of objects, but instead we think of the composition law as the key thing. So a group is a special kind of (small) category.

Example 1.5 (Monoids as categories) A monoid is precisely a category with one object. Recall that a **monoid** is a set together with an associative and unital multiplication (but which need not have inverses).

Example 1.6 (Posets as categories) Let $(P, \leq)$ be a partially ordered (or even preordered) set (i.e. poset). Then P can be regarded as a (small) category, where the objects are the elements $p \in P$, and

$$\mathrm{Hom}_P(p, q) = \begin{cases} * & \text{if } p \leq q \\ \emptyset & \text{otherwise} \end{cases}$$

There is, however, a major difference between category theory and set theory. There is **nothing** in the language of categories that lets one look *inside* an object. We think of vector spaces having elements, spaces having points, etc. By contrast, categories treat these kinds of things as invisible. There is nothing "inside" of an object $X \in \mathcal{C}$; the only way to understand X is to understand the ways one can map into and out of X. Even if one is working with a category of "structured sets," the underlying set of an object in this category is not part of the categorical data. However, there are instances in which the "underlying set" can be recovered as a Hom-set.

[1] Feel free to omit if the notion of Lie algebra is unfamiliar.

Example 1.7 In the category **Top** of topological spaces, one can in fact recover the "underlying set" of a topological space via the hom-sets. Namely, for each topological space, the points of X are the same thing as the mappings from a one-point space into X. That is, we have

$$|X| = \mathrm{Hom}_{\mathbf{Top}}(*, X),$$

where $*$ is the one-point space.

Later we will say that the functor assigning to each space its underlying set is *corepresentable.*

Example 1.8 Let **Ab** be the category of abelian groups and group-homomorphisms. Again, the claim is that using only this category, one can recover the underlying set of a given abelian group A. This is because the elements of A can be canonically identified with *morphisms* $\mathbb{Z} \to A$ (based on where $1 \in \mathbb{Z}$ maps).

Definition 1.9 We say that $\mathcal{C}$ is a **subcategory** of the category $\mathcal{D}$ if the collection of objects of $\mathcal{C}$ is a subclass of the collection of objects of $\mathcal{D}$, and if whenever $X, Y \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \subset \mathrm{Hom}_{\mathcal{D}}(X, Y)$$

with the laws of composition in $\mathcal{C}$ induced by that in $\mathcal{D}$.

$\mathcal{C}$ is called a **full subcategory** if $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(X, Y)$ whenever $X, Y \in \mathcal{C}$.

Example 1.10 The category of abelian groups is a full subcategory of the category of groups.

1.2 The language of commutative diagrams

While the language of categories is, of course, purely algebraic, it will be convenient for psychological reasons to visualize categorical arguments through diagrams. We shall introduce this notation here.

Let $\mathcal{C}$ be a category, and let X, Y be objects in $\mathcal{C}$. If $f \in \mathrm{Hom}(X, Y)$, we shall sometimes write f as an arrow

$$f : X \to Y$$

or

$$X \stackrel{f}{\to} Y$$

as if f were an actual function. If $X \stackrel{f}{\to} Y$ and $Y \stackrel{g}{\to} Z$ are morphisms, composition $g \circ f : X \to Z$ can be visualized by the picture

$$X \stackrel{f}{\to} Y \stackrel{g}{\to} Z.$$

Finally, when we work with several objects, we shall often draw collections of morphisms into diagrams, where arrows indicate morphisms between two objects.

Definition 1.11 A diagram will be said to **commute** if whenever one goes from one object in the diagram to another by following the arrows in the right order, one obtains the same morphism. For instance, the commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{f'} & W \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{g'} & Z \end{array}$$

is equivalent to the assertion that

$$g \circ f' = g' \circ f \in \mathrm{Hom}(X, Z).$$

As an example, the assertion that the associative law holds in a category $\mathcal{C}$ can be stated as follows. For every quadruple $X, Y, Z, W \in \mathcal{C}$, the following diagram (of *sets*) commutes:

$$\begin{array}{ccc} \mathrm{Hom}(X,Y)\times \mathrm{Hom}(Y,Z)\times \mathrm{Hom}(Z,W) & \longrightarrow & \mathrm{Hom}(X,Z)\times \mathrm{Hom}(Z,W) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(X,Y)\times \mathrm{Hom}(Y,W) & \longrightarrow & \mathrm{Hom}(X,W). \end{array}$$

Here the maps are all given by the composition laws in $\mathcal{C}$. For instance, the downward map to the left is the product of the identity on $\mathrm{Hom}(X,Y)$ with the composition law $\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(Z,W) \to \mathrm{Hom}(Y,W)$.

1.3 Isomorphisms

Classically, one can define an isomorphism of groups as a bijection that preserves the group structure. This does not generalize well to categories, as we do not have a notion of "bijection," as there is no way (in general) to talk about the "underlying set" of an object. Moreover, this definition does not generalize well to topological spaces: there, an isomorphism should not just be a bijection, but something which preserves the topology (in a strong sense), i.e. a homeomorphism.

Thus we make:

Definition 1.12 An **isomorphism** between objects X, Y in a category $\mathcal{C}$ is a map $f : X \to Y$ such that there exists $g : Y \to X$ with

$$g \circ f = 1_X, \quad f \circ g = 1_Y.$$

Such a g is called an **inverse** to f.

Remark It is easy to check that the inverse g is unique. Indeed, suppose g, g' both were inverses to f. Then

$$g' = g' \circ 1_Y = g' \circ (f \circ g) = (g' \circ f) \circ g = 1_X \circ g = g.$$

This notion is isomorphism is more correct than the idea of being one-to-one and onto. A bijection of topological spaces is not necessarily a homeomorphism.

Example 1.13 It is easy to check that an isomorphism in the category **Grp** is an isomorphism of groups, that an isomorphism in the category **Set** is a bijection, and so on.

We are supposed to be able to identify isomorphic objects. In the categorical sense, this means mapping into X should be the same as mapping into Y, if X, Y are isomorphic, via an isomorphism $f : X \to Y$. Indeed, let Z be another object of $\mathcal{C}$. Then we can define a map

$$\mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$$

given by post-composition with f. This is a *bijection* if f is an isomorphism (the inverse is given by postcomposition with the inverse to f). Similarly, one can easily see that mapping *out of* X is essentially the same as mapping out of Y. Anything in general category theory that is true for X should be true for Y (as general category theory can only try to understand X in terms of maps into or out of it!).

Exercise 0.1 The relation "X, Y are isomorphic" is an equivalence relation on the class of objects of a category $\mathcal{C}$.

Exercise 0.2 Let P be a preordered set, and make P into a category as in Example 1.6. Then P is a poset if and only if two isomorphic objects are equal.

For the next exercise, we need:

Definition 1.14 A **groupoid** is a category where every morphism is an isomorphism.

Exercise 0.3 The sets $\mathrm{Hom}_{\mathcal{C}}(A, A)$ are *groups* if $\mathcal{C}$ is a groupoid and $A \in \mathcal{C}$. A group is essentially the same as a groupoid with one object.

Exercise 0.4 Show that the following is a groupoid. Let X be a topological space, and let $\Pi_1(X)$ be the category defined as follows: the objects are elements of X, and morphisms $x \to y$ (for $x, y \in X$) are homotopy classes of maps $[0, 1] \to X$ (i.e. paths) that send $0 \mapsto x, 1 \mapsto y$. Composition of maps is given by concatenation of paths. (Check that, because one is working with *homotopy classes* of paths, composition is associative.)

$\Pi_1(X)$ is called the **fundamental groupoid** of X. Note that $\mathrm{Hom}_{\Pi_1(X)}(x, x)$ is the **fundamental group** $\pi_1(X, x)$.

§2 Functors

A functor is a way of mapping from one category to another: each object is sent to another object, and each morphism is sent to another morphism. We shall study many functors in the sequel: localization, the tensor product, Hom, and fancier ones like $\mathrm{Tor}, \mathrm{Ext}$, and local cohomology functors. The main benefit of a functor is that it doesn't simply send objects to other objects, but also morphisms to morphisms: this allows one to get new commutative diagrams from old ones. This will turn out to be a powerful tool.

2.1 Covariant functors

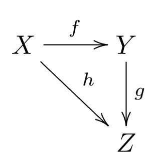
Let $\mathcal{C}, \mathcal{D}$ be categories. If $\mathcal{C}, \mathcal{D}$ are categories of structured sets (of possibly different types), there may be a way to associate objects in $\mathcal{D}$ to objects in $\mathcal{C}$. For instance, to every group G we can associate its *group ring* $\mathbb{Z}[G]$ (which we do not define here); to each topological space we can associate its *singular chain complex*, and so on. In many cases, given a map between objects in $\mathcal{C}$ preserving the relevant structure, there will be an induced map on the corresponding objects in $\mathcal{D}$. It is from here that we define a *functor.*

Definition 2.1 A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of a function $F : \mathcal{C} \to \mathcal{D}$ (that is, a rule that assigns to each object in $\mathcal{C}$ an object of $\mathcal{D}$) and, for each pair $X, Y \in \mathcal{C}$, a map $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$, which preserves the identity maps and composition.

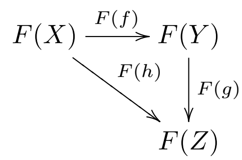
In detail, the last two conditions state the following.

1. If $X \in \mathcal{C}$, then $F(1_X)$ is the identity morphism $1_{F(X)} : F(X) \to F(X)$.
2. If $A \stackrel{f}{\to} B \stackrel{g}{\to} C$ are morphisms in $\mathcal{C}$, then $F(g \circ f) = F(g) \circ F(f)$ as morphisms $F(A) \to F(C)$. Alternatively, we can say that F *preserves commutative diagrams.*

In the last statement of the definition, note that if



is a commutative diagram in $\mathcal{C}$, then the diagram obtained by applying the functor F, namely



also commutes. It follows that applying F to more complicated commutative diagrams also yields new commutative diagrams.

Let us give a few examples of functors.

Example 2.2 There is a functor from **Sets** $\to$ **AbelianGrp** sending a set S to the free abelian group on the set. (For the definition of a free abelian group, or more generally a free R-module over a ring R, see **??**.)

Example 2.3 Let X be a topological space. Then to it we can associate the set $\pi_0(X)$ of *connected components* of X.

Recall that the continuous image of a connected set is connected, so if $f : X \to Y$ is a continuous map and $X' \subset X$ connected, $f(X')$ is contained in a connected component of Y. It follows that π_0 is a functor **Top** $\to$ **Sets**. In fact, it is a functor on the *homotopy category* as well, because homotopic maps induce the same maps on π_0.

Example 2.4 Fix n. There is a functor from **Top** $\to$ **AbGrp** (categories of topological spaces and abelian groups) sending a space X to its nth homology group $H_n(X)$. We know that given a map of spaces $f : X \to Y$, we get a map of abelian groups $f_* : H_n(X) \to H_n(Y)$. See [Hat02], for instance.

We shall often need to compose functors. For instance, we will want to see, for instance, that the *tensor product* (to be defined later, see **??**) is associative, which is really a statement about composing functors. The following (mostly self-explanatory) definition elucidates this.

Definition 2.5 If $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories, $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{E}$ are covariant functors, then one can define a **composite functor**

$$F \circ G : \mathcal{C} \to \mathcal{E}$$

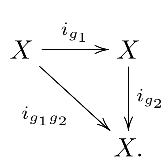
This sends an object $X \in \mathcal{C}$ to $G(F(X))$. Similarly, a morphism $f : X \to Y$ is sent to $G(F(f)) : G(F(X)) \to G(F(Y))$. We leave the reader to check that this is well-defined.

Example 2.6 In fact, because we can compose functors, there is a *category of categories.* Let **Cat** have objects as the small categories, and morphisms as functors. Composition is defined as in Definition 2.5.

Example 2.7 (Group actions) Fix a group G. Let us understand what a functor $B_G \to$ **Sets** is. Here B_G is the category of Example 1.4.

The unique object $*$ of B_G goes to some set X. For each element $g \in G$, we get a map $g : * \to *$ and thus a map $X \to X$. This is supposed to preserve the composition law (which in G is just multiplication), as well as identities.

In particular, we get maps $i_g : X \to X$ corresponding to each $g \in G$, such that the following diagram commutes for each $g_1, g_2 \in G$:



Moreover, if $e \in G$ is the identity, then $i_e = 1_X$. So a functor $B_G \to \mathbf{Sets}$ is just a left G-action on a set X.

An important example of functors is given by the following. Let $\mathcal{C}$ be a category of "structured sets." Then, there is a functor $F : \mathcal{C} \to \mathbf{Sets}$ that sends a structured set to the underlying set. For instance, there is a functor from groups to sets that forgets the group structure. More generally, suppose given two categories $\mathcal{C}, \mathcal{D}$, such that $\mathcal{C}$ can be regarded as "structured objects in $\mathcal{D}$." Then there is a functor $\mathcal{C} \to \mathcal{D}$ that forgets the structure. Such examples are called *forgetful functors.*

2.2 Contravariant functors

Sometimes what we have described above are called *covariant functors*. Indeed, we shall also be interested in similar objects that reverse the arrows, such as duality functors:

Definition 2.8 A **contravariant functor** $\mathcal{C} \stackrel{F}{\to} \mathcal{D}$ (between categories $\mathcal{C}, \mathcal{D}$) is similar data as in Definition 2.1 except that now a map $X \stackrel{f}{\to} Y$ now goes to a map $F(Y) \stackrel{F(f)}{\to} F(X)$. Composites are required to be preserved, albeit in the other direction. In other words, if $X \stackrel{f}{\to} Y, Y \stackrel{g}{\to} Z$ are morphisms, then we require

$$F(g \circ f) = F(f) \circ F(g) : F(Z) \to F(X).$$

We shall sometimes say just "functor" for *covariant functor*. When we are dealing with a contravariant functor, we will always say the word "contravariant."

A contravariant functor also preserves commutative diagrams, except that the arrows have to be reversed. For instance, if $F : \mathcal{C} \to \mathcal{D}$ is contravariant and the diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & \nearrow & \\ B & & \end{array}$$

is commutative in $\mathcal{C}$, then the diagram

$$\begin{array}{ccc} F(A) & \longleftarrow & F(C) \\ \uparrow & \swarrow & \\ F(B) & & \end{array}$$

commutes in $\mathcal{D}$.

One can, of course, compose contravariant functors as in Definition 2.5. But the composition of two contravariant functors will be *covariant.* So there is no "category of categories" where the morphisms between categories are contravariant functors.

Similarly as in Example 2.7, we have:

Example 2.9 A **contravariant** functor from B_G (defined as in Example 1.4) to **Sets** corresponds to a set with a *right* G-action.

Example 2.10 (Singular cohomology) In algebraic topology, one encounters contravariant functors on the homotopy category of topological spaces via the *singular cohomology* functors $X \mapsto H^n(X; \mathbb{Z})$. Given a continuous map $f : X \to Y$, there is a homomorphism of groups

$$f^* : H^n(Y; \mathbb{Z}) \to H^n(X; \mathbb{Z}).$$

Example 2.11 (Duality for vector spaces) On the category **Vect** of vector spaces over a field k, we have the contravariant functor

$$V \mapsto V^\vee.$$

sending a vector space to its dual $V^\vee = \mathrm{Hom}(V, k)$. Given a map $V \to W$ of vector spaces, there is an induced map

$$W^\vee \to V^\vee$$

given by the transpose.

Example 2.12 If we map $B_G \to B_G$ sending $* \mapsto *$ and $g \mapsto g^{-1}$, we get a contravariant functor.

We now give a useful (linguistic) device for translating between covariance and contravariance.

Definition 2.13 (The opposite category) Let $\mathcal{C}$ be a category. Define the **opposite category** $\mathcal{C}^{op}$ of $\mathcal{C}$ to have the same objects as $\mathcal{C}$ but such that the morphisms between X, Y in $\mathcal{C}^{op}$ are those between Y and X in $\mathcal{C}$.

There is a contravariant functor $\mathcal{C} \to \mathcal{C}^{op}$. In fact, contravariant functors out of $\mathcal{C}$ are the *same* as covariant functors out of $\mathcal{C}^{op}$.

As a result, when results are often stated for both covariant and contravariant functors, for instance, we can often reduce to the covariant case by using the opposite category.

Exercise 0.5 A map that is an isomorphism in $\mathcal{C}$ corresponds to an isomorphism in $\mathcal{C}^{op}$.

2.3 Functors and isomorphisms

Now we want to prove a simple and intuitive fact: if isomorphisms allow one to say that one object in a category is "essentially the same" as another, functors should be expected to preserve this.

Proposition 2.14 *If $f : X \to Y$ is a map in $\mathcal{C}$, and $F : \mathcal{C} \to \mathcal{D}$ is a functor, then $F(f) : FX \to FY$ is an isomorphism.*

The proof is quite straightforward, though there is an important point here. Note that the analogous result holds for *contravariant* functors too.

Proof. If we have maps $f : X \to Y$ and $g : Y \to X$ such that the composites both ways are identities, then we can apply the functor F to this, and we find that since

$$f \circ g = 1_Y, \quad g \circ f = 1_X,$$

it must hold that

$$F(f) \circ F(g) = 1_{F(Y)}, \quad F(g) \circ F(f) = 1_{F(X)}.$$

We have used the fact that functors preserve composition and identities. This implies that $F(f)$ is an isomorphism, with inverse $F(g)$. ▲

Categories have a way of making things so general that are trivial. Hence, this material is called general abstract nonsense. Moreover, there is another philosophical point about category theory to be made here: often, it is the definitions, and not the proofs, that matter. For instance, what matters here is not the theorem, but the *definition of an isomorphism.* It is a categorical one, and much more general than the usual notion via injectivity and surjectivity.

Example 2.15 As a simple example, $\{0, 1\}$ and $[0, 1]$ are not isomorphic in the homotopy category of topological spaces (i.e. are not homotopy equivalent) because $\pi_0([0, 1]) = *$ while $\pi_0(\{0, 1\})$ has two elements.

Example 2.16 More generally, the higher homotopy group functors π_n (see [Hat02]) can be used to show that the n-sphere S^n is not homotopy equivalent to a point. For then $\pi_n(S^n, *)$ would be trivial, and it is not.

There is room, nevertheless, for something else. Instead of having something that sends objects to other objects, one could have something that sends an object to a map.

2.4 Natural transformations

Suppose $F, G : \mathcal{C} \to \mathcal{D}$ are functors.

Definition 2.17 A **natural transformation** $T : F \to G$ consists of the following data. For each $X \in C$, there is a morphism $TX : FX \to GX$ satisfying the following condition. Whenever $f : X \to Y$ is a morphism, the following diagram must commute:

$$\begin{array}{ccc} FX & \xrightarrow{F(f)} & FY \\ \downarrow{\scriptstyle TX} & & \downarrow{\scriptstyle TY} \\ GX & \xrightarrow{G(f)} & GY \end{array} .$$

If TX is an isomorphism for each X, then we shall say that T is a **natural isomorphism.**

It is similarly possible to define the notion of a natural transformation between *contravariant* functors.

When we say that things are "natural" in the future, we will mean that the transformation between functors is natural in this sense. We shall use this language to state theorems conveniently.

Example 2.18 (The double dual) Here is the canonical example of "naturality." Let $\mathcal{C}$ be the category of finite-dimensional vector spaces over a given field k. Let us further restrict the category such that the only morphisms are the *isomorphisms* of vector spaces. For each $V \in \mathcal{C}$, we know that there is an isomorphism

$$V \simeq V^\vee = \mathrm{Hom}_k(V, k),$$

because both have the same dimension.

Moreover, the maps $V \mapsto V, V \mapsto V^\vee$ are both covariant functors on $\mathcal{C}$.[2] The first is the identity functor; for the second, if $f : V \to W$ is an isomorphism, then there is induced a transpose map $f^t : W^\vee \to V^\vee$ (defined by sending a map $W \to k$ to the precomposition $V \stackrel{f}{\to} W \to k$), which is an isomorphism; we can take its inverse. So we have two functors from $\mathcal{C}$ to itself, the identity and the dual, and we know that $V \simeq V^\vee$ for each V (though we have not chosen any particular set of isomorphisms).

However, the isomorphism $V \simeq V^\vee$ *cannot* be made natural. That is, there is no way of choosing isomorphisms

$$T_V : V \simeq V^\vee$$

such that, whenever $f : V \to W$ is an isomorphism of vector spaces, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle T_V} & & \downarrow{\scriptstyle T_W} \\ V^\vee & \xrightarrow{(f^t)^{-1}} & W^\vee. \end{array}$$

[2]Note that the dual $\vee$ was defined as a *contravariant* functor in Example 2.11.

Indeed, fix $d > 1$, and choose $V = k^d$. Identify $V^\vee$ with k^d, and so the map T_V is a d-by-d matrix M with coefficients in k. The requirement is that for each *invertible* d-by-d matrix N, we have

$$(N^t)^{-1}M = MN,$$

by considering the above diagram with $V = W = k^d$, and f corresponding to the matrix N. This is impossible unless $M = 0$, by elementary linear algebra.

Nonetheless, it *is* possible to choose a natural isomorphism

$$V \simeq V^{\vee\vee}.$$

To do this, given V, recall that $V^{\vee\vee}$ is the collection of maps $V^\vee \to k$. To give a map $V \to V^{\vee\vee}$ is thus the same as giving linear functions $l_v, v \in V$ such that $l_v : V \to k$ is linear in v. We can do this by letting l_v be "evaluation at v." That is, l_v sends a linear functional $\ell : V \to k$ to $\ell(v) \in k$. We leave it to the reader to check (easily) that this defines a homomorphism $V \to V^{\vee\vee}$, and that everything is natural.

Exercise 0.6 Suppose there are two functors $B_G \to \mathbf{Sets}$, i.e. G-sets. What is a natural transformation between them?

Natural transformations can be *composed*. Suppose given functors $F, G, H : \mathcal{C} \to \mathcal{D}$ a natural transformation $T : F \to G$ and a natural transformation $U : G \to H$. Then, for each $X \in \mathcal{C}$, we have maps $TX : FX \to GX, UX : GX \to HY$. We can compose U with T to get a natural transformation $U \circ T : F \to H$.

In fact, we can thus define a *category* of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ (at least if $\mathcal{C}, \mathcal{D}$ are small). The objects of this category are the functors $F : \mathcal{C} \to \mathcal{D}$. The morphisms are natural transformations between functors. Composition of morphisms is as above.

2.5 Equivalences of categories

Often we want to say that two categories $\mathcal{C}, \mathcal{D}$ are "essentially the same." One way of formulating this precisely is to say that $\mathcal{C}, \mathcal{D}$ are *isomorphic* in the category of categories. Unwinding the definitions, this means that there exist functors

$$F : \mathcal{C} \to \mathcal{D}, \quad G : \mathcal{D} \to \mathcal{C}$$

such that $F \circ G = 1_{\mathcal{D}}, G \circ F = 1_{\mathcal{C}}$. This notion, of *isomorphism* of categories, is generally far too restrictive.

For instance, we could consider the category of all finite-dimensional vector spaces over a given field k, and we could consider the full subcategory of vector spaces of the form k^n. Clearly both categories encode essentially the same mathematics, in some sense, but they are not isomorphic: one has a countable set of objects, while the other has an uncountable set of objects. Thus, we need a more refined way of saying that two categories are "essentially the same."

Definition 2.19 Two categories $\mathcal{C}, \mathcal{D}$ are called **equivalent** if there are functors

$$F : \mathcal{C} \to \mathcal{D}, \quad G : \mathcal{D} \to \mathcal{C}$$

and natural isomorphisms

$$FG \simeq 1_{\mathcal{D}}, \quad GF \simeq 1_{\mathcal{C}}.$$

For instance, the category of all vector spaces of the form k^n is equivalent to the category of all finite-dimensional vector spaces. One functor is the inclusion from vector spaces of the form k^n; the other functor maps a finite-dimensional vector space V to $k^{\dim V}$. Defining the second functor properly is, however, a little more subtle. The next criterion will be useful.

Definition 2.20 A functor $F : \mathcal{C} \to \mathcal{D}$ is **fully faithful** if $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ is a bijection for each pair of objects $X, Y \in \mathcal{C}$. F is called **essentially surjective** if every element of $\mathcal{D}$ is isomorphic to an object in the image of F.

So, for instance, the inclusion of a full subcategory is fully faithful (by definition). The forgetful functor from groups to sets is not fully faithful, because not all functions between groups are automatically homomorphisms.

Proposition 2.21 *A functor $F : \mathcal{C} \to \mathcal{D}$ induces an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. **TO BE ADDED:** this proof, and the definitions in the statement. ▲

§3 Various universal constructions

Now that we have introduced the idea of a category and showed that a functor takes isomorphisms to isomorphisms, we shall take various steps to characterize objects in terms of maps (the most complete of which is the Yoneda lemma, Theorem 4.2). In general category theory, this is generally all we *can* do, since this is all the data we are given. We shall describe objects satisfying certain "universal properties" here.

As motivation, we first discuss the concept of the "product" in terms of a universal property.

3.1 Products

Recall that if we have two sets X and Y, the product $X \times Y$ is the set of all elements of the form (x, y) where $x \in X$ and $y \in Y$. The product is also equipped with natural projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ that take (x, y) to x and y respectively. Thus any element of $X \times Y$ is uniquely determined by where they project to on X and Y. In fact, this is the case more generally; if we have an index set I and a product $X = \prod_{i \in I} X_i$, then an element $x \in X$ determined uniquely by where where the projections $p_i(x)$ land in X_i.

To get into the categorical spirit, we should speak not of elements but of maps to X. Here is the general observation: if we have any other set S with maps $f_i : S \to X_i$ then there is a unique map $S \to X = \prod_{i \in I} X_i$ given by sending $s \in S$ to the element $\{f_i(s)\}_{i \in I}$. This leads to the following characterization of a product using only "mapping properties."

Definition 3.1 Let $\{X_i\}_{i \in I}$ be a collection of objects in some category $\mathcal{C}$. Then an object $P \in \mathcal{C}$ with projections $p_i : P \to X_i$ is said to be the **product** $\prod_{i \in I} X_i$ if the following "universal property" holds: let S be any other object in $\mathcal{C}$ with maps $f_i : S \to X_i$. Then there is a unique morphism $f : S \to P$ such that $p_i f = f_i$.

In other words, to map into X is the same as mapping into all the $\{X_i\}$ at once. We have thus given a precise description of how to map into X. Note that, however, the product need not exist! If it does, however, we can express the above formalism by the following natural isomorphism of contravariant functors

$$\mathrm{Hom}(\cdot, \prod_I X_i) \simeq \prod_I \mathrm{Hom}(\cdot, X_i).$$

This is precisely the meaning of the last part of the definition. Note that this observation shows that products in the category of *sets* are really fundamental to the idea of products in any category.

Example 3.2 One of the benefits of this construction is that an actual category is not specified; thus when we take $\mathcal{C}$ to be **Sets**, we recover the cartesian product notion of sets, but if we take $\mathcal{C}$ to be **Grp**, we achieve the regular notion of the product of groups (the reader is invited to check these statements).

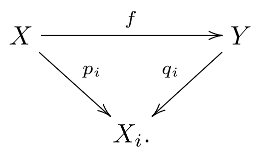
The categorical product is not unique, but it is as close to being so as possible.

Proposition 3.3 (Uniqueness of products) *Any two products of the collection $\{X_i\}$ in $\mathcal{C}$ are isomorphic by a unique isomorphism commuting with the projections.*

This is a special case of a general "abstract nonsense" type result that we shall see many more of in the sequel. The precise statement is the following: let X be a product of the $\{X_i\}$ with projections $p_i : X \to X_i$, and let Y be a product of them too, with projections $q_i : Y \to X_i$. Then the claim is that there is a *unique* isomorphism

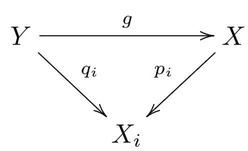
$$f : X \to Y$$

such that the diagrams below commute for each $i \in I$:



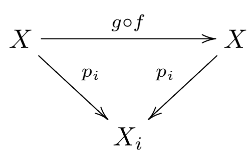
(1)

Proof. This is a "trivial" result, and is part of a general fact that objects with the same universal property are always canonically isomorphic. Indeed, note that the projections $p_i : X \to X_i$ and the fact that mapping into Y is the same as mapping into all the X_i gives a unique map $f : X \to Y$ making the diagrams (1) commute. The same reasoning (applied to the $q_i : Y \to X_i$) gives a map $g : Y \to X$ making the diagrams



(2)

commute. By piecing the two diagrams together, it follows that the composite $g \circ f$ makes the diagram



(3)

commute. But the identity $1_X : X \to X$ also would make (3) commute, and the *uniqueness* assertion in the definition of the product shows that $g \circ f = 1_X$. Similarly, $f \circ g = 1_Y$. We are done. ▲

Remark If we reverse the arrows in the above construction, the universal property obtained (known as the "coproduct") characterizes disjoint unions in the category of sets and free products in the category of groups. That is, to map *out* of a coproduct of objects $\{X_i\}$ is the same as mapping out of each of these. We shall later study this construction more generally.

EXERCISE 0.7 Let P be a poset, and make P into a category as in Example 1.6. Fix $x, y \in P$. Show that the *product* of x, y is the greatest lower bound of $\{x, y\}$ (if it exists). This claim holds more generally for arbitrary subsets of P.

In particular, consider the poset of subsets of a given set S. Then the "product" in this category corresponds to the intersection of subsets.

We shall, in this section, investigate this notion of "universality" more thoroughly.

3.2 Initial and terminal objects

We now introduce another example of universality, which is simpler but more abstract than the products introduced in the previous section.

Definition 3.4 Let $\mathcal{C}$ be a category. An **initial object** in $\mathcal{C}$ is an object $X \in \mathcal{C}$ with the property that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has one element for all $Y \in \mathcal{C}$.

So there is a unique map out of X into each $Y \in \mathcal{C}$. Note that this idea is faithful to the categorical spirit of describing objects in terms of their mapping properties. Initial objects are very easy to map *out* of.

Example 3.5 If $\mathcal{C}$ is **Sets**, then the empty set $\emptyset$ is an initial object. There is a unique map from the empty set into any other set; one has to make no decisions about where elements are to map when constructing a map $\emptyset \to X$.

Example 3.6 In the category **Grp** of groups, the group consisting of one element is an initial object.

Note that the initial object in **Grp** is *not* that in **Sets**. This should not be too surprising, because $\emptyset$ cannot be a group.

Example 3.7 Let P be a poset, and make it into a category as in Example 1.6. Then it is easy to see that an initial object of P is the smallest object in P (if it exists). Note that this is equivalently the product of all the objects in P. In general, the initial object of a category is not the product of all objects in $\mathcal{C}$ (this does not even make sense for a large category).

There is a dual notion, called a *terminal object*, where every object can map into it in precisely one way.

Definition 3.8 A **terminal object** in a category $\mathcal{C}$ is an object $Y \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) = *$ for each $X \in \mathcal{C}$.

Note that an initial object in $\mathcal{C}$ is the same as a terminal object in $\mathcal{C}^{op}$, and vice versa. As a result, it suffices to prove results about initial objects, and the corresponding results for terminal objects will follow formally. But there is a fundamental difference between initial and terminal objects. Initial objects are characterized by how one maps *out of* them, while terminal objects are characterized by how one maps *into* them.

Example 3.9 The one point set is a terminal object in **Sets**.

The important thing about the next "theorems" is the conceptual framework.

Proposition 3.10 (Uniqueness of the initial (or terminal) object) *Any two initial (resp. terminal) objects in $\mathcal{C}$ are isomorphic by a unique isomorphism.*

Proof. The proof is easy. We do it for terminal objects. Say Y, Y' are terminal objects. Then $\mathrm{Hom}(Y, Y')$ and $\mathrm{Hom}(Y', Y)$ are one point sets. So there are unique maps $f : Y \to Y', g : Y' \to Y$, whose composites must be the identities: we know that $\mathrm{Hom}(Y, Y), \mathrm{Hom}(Y', Y')$ are one-point sets, so the composites have no other choice to be the identities. This means that the maps $f : Y \to Y', g : Y' \to Y$ are isomorphisms. ▲

There is a philosophical point to be made here. We have characterized an object uniquely in terms of mapping properties. We have characterized it *uniquely up to unique isomorphism,* which is really the best one can do in mathematics. Two sets are not generally the "same," but they may be isomorphic up to unique isomorphism. They are different, but the sets are isomorphic up to unique isomorphism. Note also that the argument was essentially similar to that of Proposition 3.3.

In fact, we could interpret Proposition 3.3 as a special case of Proposition 3.10. If $\mathcal{C}$ is a category and $\{X_i\}_{i \in I}$ is a family of objects in $\mathcal{C}$, then we can define a category $\mathcal{D}$ as follows. An object of $\mathcal{D}$ is the data of an object $Y \in \mathcal{C}$ and morphisms $f_i : Y \to X_i$ for all $i \in I$. A morphism between objects $(Y, \{f_i : Y \to X_i\})$ and $(Z, \{g_i : Z \to X_i\})$ is a map $Y \to Z$ making the obvious diagrams commute. Then a product $\prod X_i$ in $\mathcal{C}$ is the same thing as a terminal object in $\mathcal{D}$, as one easily checks from the definitions.

3.3 Push-outs and pull-backs

Let $\mathcal{C}$ be a category.

Now we are going to talk about more examples of universal constructions, which can all be phrased via initial or terminal objects in some category. This, therefore, is the proof for the uniqueness up to unique isomorphism of *everything* we will do in this section. Later we will present these in more generality.

Suppose we have objects $A, B, C, X \in \mathcal{C}$.

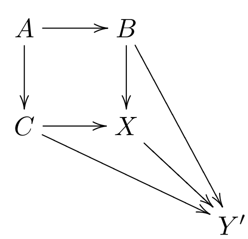
Definition 3.11 A commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & X \end{array} .$$

is a **pushout square** (and X is called the **push-out**) if, given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & Y \end{array}$$

there is a unique map $X \to Y$ making the following diagram commute:



Sometimes push-outs are also called **fibered coproducts**. We shall also write $X = C \sqcup_A B$.

In other words, to map out of $X = C \sqcup_A B$ into some object Y is to give maps $B \to Y, C \to Y$ whose restrictions to A are the same.

The next few examples will rely on notions to be introduced later.

Example 3.12 The following is a pushout square in the category of abelian groups:

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/4 \\ \downarrow & & \downarrow \\ \mathbb{Z}/6 & \longrightarrow & \mathbb{Z}/12 \end{array}$$

In the category of groups, the push-out is actually $\mathrm{SL}_2(\mathbb{Z})$, though we do not prove it. The point is that the property of a square's being a push-out is actually dependent on the category.

In general, to construct a push-out of groups $C \sqcup_A B$, one constructs the direct sum $C \oplus B$ and quotients by the subgroup generated by (a, a) (where $a \in A$ is identified with its image in $C \oplus B$). We shall discuss this later, more thoroughly, for modules over a ring.

Example 3.13 Let R be a commutative ring and let S and Q be two commutative R-algebras. In other words, suppose we have two maps of rings $s : R \to S$ and $q : R \to Q$. Then we can fit this information together into a pushout square:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ Q & \longrightarrow & X \end{array}$$

It turns out that the pushout in this case is the tensor product of algebras $S \otimes_R Q$ (see **??** for the construction). This is particularly important in algebraic geometry as the dual construction will give the correct notion of "products" in the category of "schemes" over a field.

Proposition 3.14 *Let $\mathcal{C}$ be any category. If the push-out of the diagram*

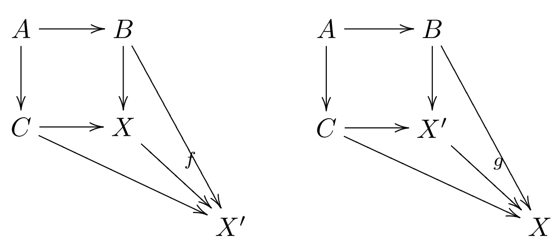
$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

exists, it is unique up to unique isomorphism.

Proof. We can prove this in two ways. One is to suppose that there were two pushout squares:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & X' \end{array}$$

Then there are unique maps $f : X \to X', g : X' \to X$ from the universal property. In detail, these maps fit into commutative diagrams



Then $g \circ f$ and $f \circ g$ are the identities of X, X' again by *uniqueness* of the map in the definition of the push-out.

Alternatively, we can phrase push-outs in terms of initial objects. We could consider the category of all diagrams as above,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array},$$

where $A \to B, A \to C$ are fixed and D varies. The morphisms in this category of diagrams consist of commutative diagrams. Then the initial object in this category is the push-out, as one easily checks. ▲

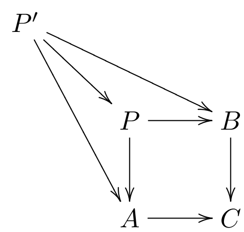
Often when studying categorical constructions, one can create a kind of "dual"construction by reversing the direction of the arrows. This is exactly the relationship between the push-out construction and the pull-back construction to be described below. So suppose we have two morphisms $A \to C$ and $B \to C$, forming a diagram

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ A & \longrightarrow & C. \end{array}$$

Definition 3.15 The **pull-back** or **fibered product** of the above diagram is an object P with two morphisms $P \to B$ and $P \to C$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

Moreover, the object P is required to be universal in the following sense: given any P' and maps $P' \to A$ and $P' \to B$ making the square commute, there is a unique map $P' \to P$ making the following diagram commute:



We shall also write $P = B \times_C A$.

Example 3.16 In the category **Set** of sets, if we have sets A, B, C with maps $f : A \to C, g : B \to C$, then the fibered product $A \times_C B$ consists of pairs $(a, b) \in A \times B$ such that $f(a) = g(b)$.

Example 3.17 (Requires prerequisites not developed yet) The next example may be omitted without loss of continuity.

As said above, the fact that the tensor product of algebras is a push-out in the category of commutative R-algebras allows for the correct notion of the "product" of schemes. We now elaborate on this example: naively one would think that we could pick the underlying space of the product scheme to just be the topological product of two Zariski topologies. However, it is an easy exercise to check that the product of two Zariski topologies in general is not Zariski! This motivates the need for a different concept.

Suppose we have a field k and two k-algebras A and B and let $X = \mathrm{Spec}(A)$and $Y = \mathrm{Spec}(B)$ be the affine k-schemes corresponding to A and B. Consider the following pull-back diagram:

$$\begin{array}{ccc} X \times_{\mathrm{Spec}(k)} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

Now, since Spec is a contravariant functor, the arrows in this pull-back diagram have been flipped; so in fact, $X \times_{\mathrm{Spec}(k)} Y$ is actually $\mathrm{Spec}(A \otimes_k B)$. This construction is motivated by the following example: let $A = k[x]$ and $B = k[y]$. Then $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are both affine lines $\mathbb{A}^1_k$ so we want a suitable notion of product that makes the product of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ the affine plane. The pull-back construction is the correct one since $\mathrm{Spec}(A) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(B) = \mathrm{Spec}(A \otimes_k B) = \mathrm{Spec}(k[x, y]) = \mathbb{A}^2_k$.

3.4 Colimits

We now want to generalize the push-out. Instead of a shape with A, B, C, we do something more general. Start with a small category I: recall that *smallness* means that the objects of I form a set. I is to be called the **indexing category**. One is supposed to picture is that I is something like the category

$$\begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array}$$

or the category

$$* \rightrightarrows *.$$

We will formulate the notion of a **colimit** which will specialize to the push-out when I is the first case.

So we will look at functors

$$F : I \to \mathcal{C},$$

which in the case of the three-element category, will just correspond to diagrams

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}.$$

We will call a **cone** on F (this is an ambiguous term) an object $X \in \mathcal{C}$ equipped with maps $F_i \to X, \forall i \in I$ such that for all maps $i \to i' \in I$, the diagram below commutes:

$$\begin{array}{ccc} F_i & \longrightarrow & X \\ \downarrow & \nearrow & \\ F_{i'} & & \end{array}.$$

An example would be a cone on the three-element category above: then this is just a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}.$$

Definition 3.18 The **colimit** of the diagram $F : I \to \mathcal{C}$, written as $\mathrm{colim} F$ or $\mathrm{colim}_I F$ or $\varinjlim_I F$, if it exists, is a cone $F \to X$ with the property that if $F \to Y$ is any other cone, then there is a unique map $X \to Y$ making the diagram

$$\begin{array}{ccc} F & \longrightarrow & X \\ & \searrow & \downarrow \\ & & Y \end{array}$$

commute. (This means that the corresponding diagram with F_i replacing F commutes for each $i \in I$.)

We could form a category $\mathcal{D}$ where the objects are the cones $F \to X$, and the morphisms from $F \to X$ and $F \to Y$ are the maps $X \to Y$ that make all the obvious diagrams commute. In this case, it is easy to see that a *colimit* of the diagram is just an initial object in $\mathcal{D}$.

In any case, we see:

Proposition 3.19 colim$F$*, if it exists, is unique up to unique isomorphism.*

Let us go through some examples. We already looked at push-outs.

Example 3.20 Consider the category I visualized as

$$*, *, *, *.$$

So I consists of four objects with no non-identity morphisms. A functor $F : I \to \mathbf{Sets}$ is just a list of four sets A, B, C, D. The colimit is just the disjoint union $A \sqcup B \sqcup C \sqcup D$. This is the universal property of the disjoint union. To map out of the disjoint union is the same thing as mapping out of each piece.

Example 3.21 Suppose we had the same category I but the functor F took values in the category of abelian groups. Then F corresponds, again, to a list of four abelian groups. The colimit is the direct sum. Again, the direct sum is characterized by the same universal property.

Example 3.22 Suppose we had the same I $(*, *, *, *)$ the functor took its value in the category of groups. Then the colimit is the free product of the four groups.

Example 3.23 Suppose we had the same I and the category $\mathcal{C}$ was of commutative rings with unit. Then the colimit is the tensor product.

So the idea of a colimit unifies a whole bunch of constructions. Now let us take a different example.

Example 3.24 Take

$$I = * \rightrightarrows *.$$

So a functor $I \to \mathbf{Sets}$ is a diagram

$$A \rightrightarrows B.$$

Call the two maps $f, g : A \to B$. To get the colimit, we take B and mod out by the equivalence relation generated by $f(a) \sim g(a)$. To hom out of this is the same thing as homming out of B such that the pullbacks to A are the same.

This is the relation **generated** as above, not just as above. It can get tricky.

Definition 3.25 When I is just a bunch of points $*, *, *, \ldots$ with no non-identity morphisms, then the colimit over I is called the **coproduct**.

We use the coproduct to mean things like direct sums, disjoint unions, and tensor products. If $\{A_i, i \in I\}$ is a collection of objects in some category, then we find the universal property of the coproduct can be stated succinctly:

$$\mathrm{Hom}_{\mathcal{C}}(\bigsqcup_I A_i, B) = \prod \mathrm{Hom}_{\mathcal{C}}(A_i, B).$$

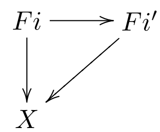
Definition 3.26 When I is $* \rightrightarrows *$, the colimit is called the **coequalizer**.

Theorem 3.27 *If $\mathcal{C}$ has all coproducts and coequalizers, then it has all colimits.*

Proof. Let $F : I \to \mathcal{C}$ be a functor, where I is a small category. We need to obtain an object X with morphisms

$$Fi \to X, \quad i \in I$$

such that for each $f : i \to i'$, the diagram below commutes:



and such that X is universal among such diagrams.

To give such a diagram, however, is equivalent to giving a collection of maps

$$Fi \to X$$

that satisfy some conditions. So X should be thought of as a quotient of the coproduct $\sqcup_i Fi$. Let us consider the coproduct $\sqcup_{i \in I, f} Fi$, where f ranges over all morphisms in the category I that start from i. We construct two maps

$$\sqcup_f Fi \rightrightarrows \sqcup_f Fi,$$

whose coequalizer will be that of F. The first map is the identity. The second map sends a factor▲

3.5 Limits

As in the example with pull-backs and push-outs and products and coproducts, one can define a limit by using the exact same universal property above just with all the arrows reversed.

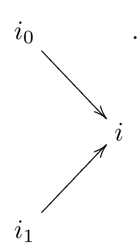
Example 3.28 The product is an example of a limit where the indexing category is a small category I with no morphisms other than the identity. This example shows the power of universal constructions; by looking at colimits and limits, a whole variety of seemingly unrelated mathematical constructions are shown to be in the same spirit.

3.6 Filtered colimits

Filtered colimits are colimits over special indexing categories I which look like totally ordered sets. These have several convenient properties as compared to general colimits. For instance, in the category of *modules* over a ring (to be studied in Chapter 1), we shall see that filtered colimits actually preserve injections and surjections. In fact, they are *exact.* This is not true in more general categories which are similarly structured.

Definition 3.29 An indexing category is **filtered** if the following hold:

1. Given $i_0, i_1 \in I$, there is a third object $i \in I$ such that both i_0, i_1 map into i. So there is a diagram



2. Given any two maps $i_0 \rightrightarrows i_1$, there exists i and $i_1 \to i$ such that the two maps $i_0 \rightrightarrows i$ are equal: intuitively, any two ways of pushing an object into another can be made into the same eventually.

Example 3.30 If I is the category

$$* \to * \to * \to \dots,$$

i.e. the category generated by the poset $\mathbb{Z}_{\geq 0}$, then that is filtered.

Example 3.31 If G is a torsion-free abelian group, the category I of finitely generated subgroups of G and inclusion maps is filtered. We don't actually need the lack of torsion.

Definition 3.32 Colimits over a filtered category are called **filtered colimits**.

Example 3.33 Any torsion-free abelian group is the filtered colimit of its finitely generated subgroups, which are free abelian groups.

This gives a simple approach for showing that a torsion-free abelian group is flat.

Proposition 3.34 *If I is filtered*[3] *and $\mathcal{C} = \mathbf{Sets}, \mathbf{Abgrp}, \mathbf{Grps}$, etc., and $F : I \to \mathcal{C}$ is a functor, then $\mathrm{colim}_I F$ exists and is given by the disjoint union of $F_i, i \in I$ modulo the relation $x \in F_i$ is equivalent to $x' \in F_{i'}$ if x maps to x' under $F_i \to F_{i'}$. This is already an equivalence relation.*

The fact that the relation given above is transitive uses the filtering of the indexing set. Otherwise, we would need to use the relation generated by it.

Example 3.35 Take $\mathbb{Q}$. This is the filtered colimit of the free submodules $\mathbb{Z}(1/n)$.

Alternatively, choose a sequence of numbers $m_1, m_2, \dots,$ such that for all p, n, we have $p^n \mid m_i$ for $i \gg 0$. Then we have a sequence of maps

$$\mathbb{Z} \stackrel{m_1}{\to} \mathbb{Z} \stackrel{m_2}{\to} \mathbb{Z} \to \dots.$$

The colimit of this is $\mathbb{Q}$. There is a quick way of seeing this, which is left to the reader.

When we have a functor $F : I \to \mathbf{Sets}, \mathbf{Grps}, \mathbf{Modules}$ taking values in a "nice" category (e.g. the category of sets, modules, etc.), one can construct the colimit by taking the union of the $F_i, i \in I$ and quotienting by the equivalence relation $x \in F_i \sim x' \in F_{i'}$ if $f : i \to i'$ sends x into x'. This is already an equivalence relation, as one can check.

Another way of saying this is that we have the disjoint union of the F_i modulo the relation that $a \in F_i$ and $b \in F_{i'}$ are equivalent if and only if there is a later i'' with maps $i \to i'', i' \to i''$ such that a, b both map to the same thing in $F_{i''}$.

One of the key properties of filtered colimits is that, in "nice" categories they commute with finite limits.

Proposition 3.36 *In the category of sets, filtered colimits and finite limits commute with each other.*

The reason this result is so important is that, as we shall see, it will imply that in categories such as the category of R-modules, filtered colimits preserve *exactness*.

[3] Some people say filtering.

Proof. Let us show that filtered colimits commute with (finite) products in the category of sets. The case of an equalizer is similar, and finite limits can be generated from products and equalizers.

So let I be a filtered category, and $\{A_i\}_{i \in I}, \{B_i\}_{i \in I}$ be functors from $I \to \mathbf{Sets}$. We want to show that

$$\varinjlim_I (A_i \times B_i) = \varinjlim_I A_i \times \varinjlim_I B_i.$$

To do this, note first that there is a map in the direction $\to$ because of the natural maps $\varinjlim_I (A_i \times B_i) \to \varinjlim_I A_i$ and $\varinjlim_I (A_i \times B_i) \to \varinjlim_I B_i$. We want to show that this is an isomorphism.

Now we can write the left side as the disjoint union $\bigsqcup_I (A_i \times B_i)$ modulo the equivalence relation that (a_i, b_i) is related to (a_j, b_j) if there exist morphisms $i \to k, j \to k$ sending $(a_i, b_i), (a_j, b_j)$ to the same object in $A_k \times B_k$. For the left side, we have to work with pairs: that is, an element of $\varinjlim_I A_i \times \varinjlim_I B_i$ consists of a pair (a_{i_1}, b_{i_2}) with two pairs $(a_{i_1}, b_{i_2}), (a_{j_1}, b_{j_2})$ equivalent if there exist morphisms $i_1, j_1 \to k_1$ and $i_2, j_2 \to k_2$ such that both have the same image in $A_{k_1} \times A_{k_2}$. It is easy to see that these amount to the same thing, because of the filtering condition: we can always modify an element of $A_i \times B_j$ to some $A_k \times B_k$ for k receiving maps from i, j. ▲

Exercise 0.8 Let A be an abelian group, $e : A \to A$ an *idempotent* operator, i.e. one such that $e^2 = e$. Show that eA can be obtained as the filtered colimit of

$$A \stackrel{e}{\to} A \stackrel{e}{\to} A \dots.$$

3.7 The initial object theorem

We now prove a fairly nontrivial result, due to Freyd. This gives a sufficient condition for the existence of initial objects. We shall use it in proving the adjoint functor theorem below.

Let $\mathcal{C}$ be a category. Then we recall that $A \in \mathcal{C}$ if for each $X \in \mathcal{C}$, there is a *unique* $A \to X$. Let us consider the weaker condition that for each $X \in \mathcal{C}$, there exists *a* map $A \to X$.

Definition 3.37 Suppose $\mathcal{C}$ has equalizers. If $A \in \mathcal{C}$ is such that $\mathrm{Hom}_{\mathcal{C}}(A, X) \neq \emptyset$ for each $X \in \mathcal{C}$, then X is called **weakly initial.**

We now want to get an initial object from a weakly initial object. To do this, note first that if A is weakly initial and B is any object with a morphism $B \to A$, then B is weakly initial too. So we are going to take our initial object to be a very small subobject of A. It is going to be so small as to guarantee the uniqueness condition of an initial object. To make it small, we equalize all endomorphisms.

Proposition 3.38 *If A is a weakly initial object in $\mathcal{C}$, then the equalizer of all endomorphisms $A \to A$ is initial for $\mathcal{C}$.*

Proof. Let A' be this equalizer; it is endowed with a morphism $A' \to A$. Then let us recall what this means. For any two endomorphisms $A \rightrightarrows A$, the two pull-backs $A' \rightrightarrows A$ are equal. Moreover, if $B \to A$ is a morphism that has this property, then B factors uniquely through A'.

Now $A' \to A$ is a morphism, so by the remarks above, A' is weakly initial: to each $X \in \mathcal{C}$, there exists a morphism $A' \to X$. However, we need to show that it is unique.

So suppose given two maps $f, g : A' \rightrightarrows X$. We are going to show that they are equal. If not, consider their equalizer O. Then we have a morphism $O \to A'$ such that the post-compositions with f, g are equal. But by weak initialness, there is a map $A \to O$; thus we get a composite

$$A \to O \to A'.$$

We claim that this is a *section* of the embedding $A' \to A$. This will prove the result. Indeed, we will have constructed a section $A \to A'$, and since it factors through O, the two maps

$$A \to O \to A' \rightrightarrows X$$

are equal. Thus, composing each of these with the inclusion $A' \to A$ shows that f, g were equal in the first place.

Thus we are reduced to proving:

Lemma 3.39 *Let A be an object of a category $\mathcal{C}$. Let A' be the equalizer of all endomorphisms of A. Then any morphism $A \to A'$ is a section of the inclusion $A' \to A$.*

Proof. Consider the canonical inclusion $i : A' \to A$. We are given some map $s : A \to A'$; we must show that $si = 1_{A'}$. Indeed, consider the composition

$$A' \stackrel{i}{\to} A \stackrel{s}{\to} A' \stackrel{i}{\to} A.$$

Now i equalizes endomorphisms of A; in particular, this composition is the same as

$$A' \stackrel{i}{\to} A \stackrel{\mathrm{id}}{\to} A; \qquad \blacktriangle$$

that is, it equals i. So the map $si : A' \to A$ has the property that $isi = i$ as maps $A' \to A$. But i being a monomorphism, it follows that $si = 1_{A'}$. ▲

Theorem 3.40 (Freyd) *Let $\mathcal{C}$ be a category admitting all small limits.*[4] *Then $\mathcal{C}$ has an initial object if and only if the following **solution set condition holds:** there is a set $\{X_i, i \in I\}$ of objects in $\mathcal{C}$ such that any $X \in \mathcal{C}$ can be mapped into by one of these.*

The idea is that the family $\{X_i\}$ is somehow weakly universal *together.*

Proof. If $\mathcal{C}$ has an initial object, we may just consider that as the family $\{X_i\}$: we can hom out (uniquely!) from a universal object into anything, or in other words a universal object is weakly universal.

Suppose we have a "weakly universal family" $\{X_i\}$. Then the product $\prod X_i$ is weakly universal. Indeed, if $X \in \mathcal{C}$, choose some i' and a morphism $X_{i'} \to X$ by the hypothesis. Then this map composed with the projection from the product gives a map $\prod X_i \to X_{i'} \to X$. Proposition 3.38 now implies that $\mathcal{C}$ has an initial object. ▲

3.8 Completeness and cocompleteness

Definition 3.41 A category $\mathcal{C}$ is said to be **complete** if for every functor $F : I \to \mathcal{C}$ where I is a small category, the limit $\lim F$ exists (i.e. $\mathcal{C}$ has all small limits). If all colimits exist, then $\mathcal{C}$ is said to be **cocomplete**.

If a category is complete, various nice properties hold.

Proposition 3.42 *If $\mathcal{C}$ is a complete category, the following conditions are true:*

1. *all (finite) products exist*
2. *all pull-backs exist*
3. *there is a terminal object*

Proof. The proof of the first two properties is trivial since they can all be expressed as limits; for the proof of the existence of a terminal object, consider the empty diagram $F : \emptyset \to \mathcal{C}$. Then the terminal object is just $\lim F$. ▲

Of course, if one dualizes everything we get a theorem about cocomplete categories which is proved in essentially the same manner. More is true however; it turns out that finite (co)completeness are equivalent to the properties above if one requires the finiteness condition for the existence of (co)products.

[4] We shall later call such a category **complete**.

3.9 Continuous and cocontinuous functors

3.10 Monomorphisms and epimorphisms

We now wish to characterize monomorphisms and epimorphisms in a purely categorical setting. In categories where there is an underlying set the notions of injectivity and surjectivity makes sense but in category theory, one does not in a sense have "access" to the internal structure of objects. In this light, we make the following definition.

Definition 3.43 A morphism $f : X \to Y$ is a **monomorphism** if for any two morphisms $g_1 : X' \to X$ and $g_2 : X' \to X$, we have that $fg_1 = fg_2$ implies $g_1 = g_2$. A morphism $f : X \to Y$ is an **epimorphism** if for any two maps $g_1 : Y \to Y'$ and $g_2 : Y \to Y'$, we have that $g_1 f = g_2 f$ implies $g_1 = g_2$.

So $f : X \to Y$ is a monomorphism if whenever X' is another object in $\mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(X', X) \to \mathrm{Hom}_{\mathcal{C}}(X', Y)$$

is an injection (of sets). Epimorphisms in a category are defined similarly; note that neither definition makes any reference to *surjections* of sets.

The reader can easily check:

Proposition 3.44 *The composite of two monomorphisms is a monomorphism, as is the composite of two epimorphisms.*

Exercise 0.9 Prove Proposition 3.44.

Exercise 0.10 The notion of "monomorphism" can be detected using only the notions of fibered product and isomorphism. To see this, suppose $i : X \to Y$ is a monomorphism. Show that the diagonal

$$X \to X \times_Y X$$

is an isomorphism. (The diagonal map is such that the two projections to X both give the identity.) Conversely, show that if $i : X \to Y$ is any morphism such that the above diagonal map is an isomorphism, then i is a monomorphism.

Deduce the following consequence: if $F : \mathcal{C} \to \mathcal{D}$ is a functor that commutes with fibered products, then F takes monomorphisms to monomorphisms.

§4 Yoneda's lemma

TO BE ADDED: this section is barely fleshed out

Let $\mathcal{C}$ be a category. In general, we have said that there is no way to study an object in a category other than by considering maps into and out of it. We will see that essentially everything about $X \in \mathcal{C}$ can be recovered from these hom-sets. We will thus get an embedding of $\mathcal{C}$ into a category of functors.

4.1 The functors h_X

We now use the structure of a category to construct hom functors.

Definition 4.1 Let $X \in \mathcal{C}$. We define the contravariant functor $h_X : \mathcal{C} \to$ **Sets** via

$$h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

This is, indeed, a functor. If $g : Y \to Y'$, then precomposition gives a map of sets

$$h_X(Y') \to h_X(Y), \quad f \mapsto f \circ g$$

which satisfies all the usual identities.

As a functor, h_X encodes *all* the information about how one can map into X. It turns out that one can basically recover X from h_X, though.

4.2 The Yoneda lemma

Let $X \stackrel{f}{\to} X'$ be a morphism in $\mathcal{C}$. Then for each $Y \in \mathcal{C}$, composition gives a map

$$\mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, X').$$

It is easy to see that this induces a *natural* transformation

$$h_X \to h_{X'}.$$

Thus we get a map of sets

$$\mathrm{Hom}_{\mathcal{C}}(X, X') \to \mathrm{Hom}(h_X, h_{X'}),$$

where $h_X, h_{X'}$ lie in the category of contravariant functors $\mathcal{C} \to \mathbf{Sets}$. In other words, we have defined a *covariant functor*

$$\mathcal{C} \to \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Sets}).$$

This is called the *Yoneda embedding.* The next result states that the embedding is fully faithful.

Theorem 4.2 (Yoneda's lemma) *If $X, X' \in \mathcal{C}$, then the map $\mathrm{Hom}_{\mathcal{C}}(X, X') \to \mathrm{Hom}(h_X, h_{X'})$ is a bijection. That is, every natural transformation $h_X \to h_{X'}$ arises in one and only one way from a morphism $X \to X'$.*

Theorem 4.3 (Strong Yoneda lemma)

4.3 Representable functors

We use the same notation of the preceding section: for a category $\mathcal{C}$ and $X \in \mathcal{C}$, we let h_X be the contravariant functor $\mathcal{C} \to \mathbf{Sets}$ given by $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Definition 4.4 A contravariant functor $F : \mathcal{C} \to \mathbf{Sets}$ is **representable** if it is naturally isomorphic to some h_X.

The point of a representable functor is that it can be realized as maps into a specific object. In fact, let us look at a specific feature of the functor h_X. Consider the object $\alpha \in h_X(X)$ that corresponds to the identity. Then any morphism

$$Y \to X$$

factors *uniquely* as

$$Y \to X \stackrel{\alpha}{\to} X$$

(this is completely trivial!) so that any element of $h_X(Y)$ is a $f^*(\alpha)$ for precisely one $f : Y \to X$.

Definition 4.5 Let $F : \mathcal{C} \to \mathbf{Sets}$ be a contravariant functor. A **universal object** for $\mathcal{C}$ is a pair (X, α) where $X \in \mathcal{C}, \alpha \in F(X)$ such that the following condition holds: if Y is any object and $\beta \in F(Y)$, then there is a unique $f : Y \to X$ such that α pulls back to β under f.

In other words, $\beta = f^*(\alpha)$.

So a functor has a universal object if and only if it is representable. Indeed, we just say that the identity $X \to X$ is universal for h_X, and conversely if F has a universal object (X, α), then F is naturally isomorphic to h_X (the isomorphism $h_X \simeq F$ being given by pulling back α appropriately).

The article [Vis08] by Vistoli contains a good introduction to and several examples of this theory. Here is one of them:

Example 4.6 Consider the contravariant functor $F : \mathbf{Sets} \to \mathbf{Sets}$ that sends any set S to its power set 2^S (i.e. the collection of subsets). This is a contravariant functor: if $f : S \to T$, there is a morphism

$$2^T \to 2^S, \quad T' \mapsto f^{-1}(T').$$

This is a representable functor. Indeed, the universal object can be taken as the pair

$$(\{0, 1\}, \{1\}).$$

To understand this, note that a subset S; of S determines its *characteristic function* $\chi_{S'} : S \to \{0, 1\}$ that takes the value 1 on S and 0 elsewhere. If we consider $\chi_{S'}$ as a morphism $S \to \{0, 1\}$, we see that

$$S' = \chi_{S'}^{-1}(\{1\}).$$

Moreover, the set of subsets is in natural bijection with the set of characteristic functions, which in turn are precisely *all* the maps $S \to \{0, 1\}$. From this the assertion is clear.

We shall meet some elementary criteria for the representability of contravariant functors in the next subsection. For now, we note[5] that in algebraic topology, one often works with the *homotopy category* of pointed CW complexes (where morphisms are pointed continuous maps modulo homotopy), any contravariant functor that satisfies two relatively mild conditions (a Mayer-Vietoris condition and a condition on coproducts), is automatically representable by a theorem of Brown. In particular, this implies that the singular cohomology functors $H^n(-, G)$ (with coefficients in some group G) are representable; the representing objects are the so-called Eilenberg-MacLane spaces $K(G, n)$. See [Hat02].

4.4 Limits as representable functors

TO BE ADDED:

4.5 Criteria for representability

Let $\mathcal{C}$ be a category. We saw in the previous subsection that a representable functor must send colimits to limits. We shall now see that there is a converse under certain set-theoretic conditions. For simplicity, we start by stating the result for corepresentable functors.

Theorem 4.7 ((Co)representability theorem) *Let $\mathcal{C}$ be a complete category, and let $F : \mathcal{C} \to$* **Sets** *be a covariant functor. Suppose F preserves limits and satisfies the solution set condition: there is a set of objects $\{Y_\alpha\}$ such that, for any $X \in \mathcal{C}$ and $x \in F(X)$, there is a morphism*

$$Y_\alpha \to X$$

carrying some element of $F(Y_\alpha)$ onto x.

Then F is corepresentable.

[5] The reader unfamiliar with algebraic topology may omit these remarks.

Proof. To F, we associate the following *category* $\mathcal{D}$. An object of $\mathcal{D}$ is a pair (x, X) where $x \in F(X)$ and $X \in \mathcal{C}$. A morphism between (x, X) and (y, Y) is a map

$$f : X \to Y$$

that sends x into y (via $F(f) : F(X) \to F(Y)$). It is easy to see that F is corepresentable if and only if there is an initla object in this category; this initial object is the "universal object."

We shall apply the initial object theorem, Theorem 3.40. Let us first verify that $\mathcal{D}$ is complete; this follows because $\mathcal{C}$ is and F preserves limits. So, for instance, the product of (x, X) and (y, Y) is $((x, y), X \times Y)$; here (x, y) is the element of $F(X) \times F(Y) = F(X \times Y)$. The solution set condition states that there is a weakly initial family of objects, and the initial object theorem now implies that there is an initial object. ▲

§5 Adjoint functors

According to MacLane, "Adjoint functors arise everywhere." We shall see several examples of adjoint functors in this book (such as Hom and the tensor product). The fact that a functor has an adjoint often immediately implies useful properties about it (for instance, that it commutes with either limits or colimits); this will lead, for instance, to conceptual arguments behind the right-exactness of the tensor product later on.

5.1 Definition

Suppose $\mathcal{C}, \mathcal{D}$ are categories, and let $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be (covariant) functors.

Definition 5.1 F, G are **adjoint functors** if there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(Fc, d) \simeq \mathrm{Hom}_{\mathcal{C}}(c, Gd)$$

whenever $c \in \mathcal{C}, d \in \mathcal{D}$. F is said to be the **right adjoint,** and G is the **left adjoint.**

Here "natural" means that the two quantities are supposed to be considered as functors $\mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$.

Example 5.2 There is a simple pair of adjoint functors between **Set** and **AbGrp**. Here F sends a set A to the free abelian group (see **??** for a discussion of free modules over arbitrary rings) $\mathbb{Z}[A]$, while G is the "forgetful" functor that sends an abelian group to its underlying set. Then F and G are adjoints. That is, to give a group-homomorphism

$$\mathbb{Z}[A] \to G$$

for some abelian group G is the same as giving a map of *sets*

$$A \to G.$$

This is precisely the defining property of the free abelian group.

Example 5.3 In fact, most "free" constructions are just left adjoints. For instance, recall the universal property of the free group $F(S)$ on a set S (see [Lan02]): to give a group-homomorphism $F(S) \to G$ for G any group is the same as choosing an image in G of each $s \in S$. That is,

$$\mathrm{Hom}_{\mathbf{Grp}}(F(S), G) = \mathrm{Hom}_{\mathbf{Sets}}(S, G).$$

This states that the free functor $S \mapsto F(S)$ is left adjoint to the forgetful functor from **Grp** to **Sets**.

Example 5.4 The abelianization functor $G \mapsto G^{ab} = G/[G,G]$ from **Grp** $\to$ **AbGrp** is left adjoint to the inclusion **AbGrp** $\to$ **Grp**. That is, if G is a group and A an abelian group, there is a natural correspondence between homomorphisms $G \to A$ and $G^{ab} \to A$. Note that **AbGrp** is a subcategory of **Grp** such that the inclusion admits a left adjoint; in this situation, the subcategory is called **reflective.**

5.2 Adjunctions

The fact that two functors are adjoint is encoded by a simple set of algebraic data between them. To see this, suppose $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ are adjoint functors. For any object $c \in \mathcal{C}$, we know that

$$\mathrm{Hom}_{\mathcal{D}}(Fc, Fc) \simeq \mathrm{Hom}_{\mathcal{C}}(c, GFc),$$

so that the identity morphism $Fc \to Fc$ (which is natural in c!) corresponds to a map $c \to GFc$ that is natural in c, or equivalently a natural transformation

$$\eta : 1_{\mathcal{C}} \to GF.$$

Similarly, we get a natural transformation

$$\epsilon : FG \to 1_{\mathcal{D}}$$

where the map $FGd \to d$ corresponds to the identity $Gd \to Gd$ under the adjoint correspondence. Here η is called the **unit**, and ϵ the **counit.**

These natural transformations η, ϵ are not simply arbitrary. We are, in fact, going to show that they determine the isomorphism determine the isomorphism $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \simeq \mathrm{Hom}_{\mathcal{C}}(c, Gd)$. This will be a little bit of diagram-chasing.

We know that the isomorphism $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \simeq \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ is *natural.* In fact, this is the key point. Let $\phi : Fc \to d$ be any map. Then there is a morphism $(c, Fc) \to (c, d)$ in the product category $\mathcal{C}^{op} \times \mathcal{D}$; by naturality of the adjoint isomorphism, we get a commutative square of sets

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(Fc, Fc) & \xrightarrow{\mathrm{adj}} & \mathrm{Hom}_{\mathcal{C}}(c, GFc) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle G(\phi)_*} \\ \mathrm{Hom}_{\mathcal{D}}(Fc, d) & \xrightarrow{\mathrm{adj}} & \mathrm{Hom}_{\mathcal{C}}(c, Gd) \end{array}$$

Here the mark adj indicates that the adjoint isomorphism is used. If we start with the identity 1_{Fc} and go down and right, we get the map $c \to Gd$ that corresponds under the adjoint correspondence to $Fc \to d$. However, if we go right and down, we get the natural unit map $\eta(c) : c \to GFc$ followed by $G(\phi)$.

Thus, we have a *recipe* for constructing a map $c \to Gd$ given $\phi : Fc \to d$:

Proposition 5.5 (The unit and counit determines everything) *Let (F, G) be a pair of adjoint functors with unit and counit transformations η, ϵ.*

Then given $\phi : Fc \to d$, the adjoint map $\psi : c \to Gd$ can be constructed simply as follows. Namely, we start with the unit $\eta(c) : c \to GFc$ and take

$$\psi = G(\phi) \circ \eta(c) : c \to Gd \tag{4}$$

(here $G(\phi) : GFc \to Fd$).

In the same way, if we are given $\psi : c \to Gd$ and want to construct a map $\phi : Fc \to d$, we construct

$$\epsilon(d) \circ F(\psi) : Fc \to FGd \to d. \tag{5}$$

In particular, we have seen that the *unit and counit morphisms determine the adjoint isomorphisms.*

Since the adjoint isomorphisms $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \to \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ and $\mathrm{Hom}_{\mathcal{C}}(c, Gd) \to \mathrm{Hom}_{\mathcal{D}}(Fc, d)$ are (by definition) inverse to each other, we can determine conditions on the units and counits.

For instance, the natural transformation $F \circ \eta$ gives a natural transformation $F \circ \eta : F \to FGF$, while the natural transformation $\epsilon \circ F$ gives a natural transformation $FGF \to F$. (These are slightly different forms of composition!)

Lemma 5.6 *The composite natural transformation $F \to F$ given by $(\epsilon \circ F) \circ (F \circ \eta)$ is the identity. Similarly, the composite natural transformation $G \to GFG \to G$ given by $(G \circ \epsilon) \circ (\eta \circ G)$ is the identity.*

Proof. We prove the first assertion; the second is similar. Given $\phi : Fc \to d$, we know that we must get back to ϕ applying the two constructions above. The first step (going to a map $\psi : c \to Gd$) is by (4) $\psi = G(\phi) \circ \eta(c)$; the second step sends ψ to $\epsilon(d) \circ F(\psi)$, by (5). It follows that

$$\phi = \epsilon(d) \circ F(G(\phi) \circ \eta(c)) = \epsilon(d) \circ F(G(\phi)) \circ F(\eta(c)).$$

Now suppose we take $d = Fc$ and $\phi : Fc \to Fc$ to be the identity. We find that $F(G(\phi))$ is the identity $FGFc \to FGFc$, and consequently we find

$$\mathrm{id}_{F(c)} = \epsilon(Fc) \circ F(\eta(c)).$$

This proves the claim. ▲

Definition 5.7 Let $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be covariant functors. An **adjunction** is the data of two natural transformations

$$\eta : 1 \to GF, \quad \epsilon : FG \to 1,$$

called the **unit** and **counit**, respectively, such that the composites $(\epsilon \circ F) \circ (F \circ \epsilon) : F \to F$ and $(G \circ \epsilon) \circ (\eta \circ G)$ are the identity (that is, the identity natural transformations of F, G).

We have seen that a pair of adjoint functors gives rise to an adjunction. Conversely, an adjunction between F, G ensures that F, G are adjoint, as one may check: one uses the same formulas (4) and (5) to define the natural isomorphism.

For any set S, let $F(S)$ be the free group on S. So, for instance, the fact that there is a natural map of sets $S \to F(S)$, for any set S, and a natural map of groups $F(G) \to G$ for any group G, determines the adjunction between the free group functor from **Sets** to **Grp**, and the forgetful functor **Grp** $\to$ **Sets**.

As another example, we give a criterion for a functor in an adjunction to be fully faithful.

Proposition 5.8 *Let F, G be a pair of adjoint functors between categories $\mathcal{C}, \mathcal{D}$. Then G is fully faithful if and only if the unit maps $\eta : 1 \to GF$ are isomorphisms.*

Proof. We use the recipe (4). Namely, we have a map $\mathrm{Hom}_{\mathcal{D}}(Fc, d) \to \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ given by $\phi \mapsto G(\phi) \circ \eta(c)$. This is an isomorphism, since we have an adjunction. As a result, composition with η is an isomorphism of hom-sets if and only if $\phi \mapsto G(\phi)$ is an isomorphism. From this the result is easy to deduce. ▲

Example 5.9 For instance, recall that the inclusion functor from **AbGrp** to **Grp** is fully faithful (clear). This is a right adjoint to the abelianization functor $G \mapsto G^{ab}$. As a result, we would expect the unit map of the adjunction to be an isomorphism, by Proposition 5.8.

The unit map sends an abelian group to its abelianization: this is obviously an isomorphism, as abelianizing an abelian group does nothing.

5.3 Adjoints and (co)limits

One very pleasant property of functors that are left (resp. right) adjoints is that they preserve all colimits (resp. limits).

Proposition 5.10 *A left adjoint $F : \mathcal{C} \to \mathcal{D}$ preserves colimits. A right adjoint $G : \mathcal{D} \to \mathcal{C}$ preserves limits.*

As an example, the free functor from **Sets** to **AbGrp** is a left adjoint, so it preserves colimits. For instance, it preserves coproducts. This corresponds to the fact that if A_1, A_2 are sets, then $\mathbb{Z}[A_1 \sqcup A_2]$ is naturally isomorphic to $\mathbb{Z}[A_1] \oplus \mathbb{Z}[A_2]$.

Proof. Indeed, this is mostly formal. Let $F : \mathcal{C} \to \mathcal{D}$ be a left adjoint functor, with right adjoint G. Let $f : I \to \mathcal{C}$ be a "diagram" where I is a small category. Suppose $\mathrm{colim}_I f$ exists as an object of $\mathcal{C}$. The result states that $\mathrm{colim}_I F \circ f$ exists as an object of $\mathcal{D}$ and can be computed as $F(\mathrm{colim}_I f)$. To see this, we need to show that mapping out of $F(\mathrm{colim}_I f)$ is what we want—that is, mapping out of $F(\mathrm{colim}_I f)$ into some $d \in \mathcal{D}$—amounts to giving compatible $F(f(i)) \to d$ for each $i \in I$. In other words, we need to show that $\mathrm{Hom}_{\mathcal{D}}(F(\mathrm{colim}_I f), d) = \lim_I \mathrm{Hom}_{\mathcal{D}}(F(f(i)), d)$; this is precisely the defining property of the colimit.

But we have

$$\mathrm{Hom}_{\mathcal{D}}(F(\mathrm{colim}_I f), d) = \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_I f, Gd) = \lim_I \mathrm{Hom}_{\mathcal{C}}(fi, Gd) = \lim_I \mathrm{Hom}_{\mathcal{D}}(F(fi), d),$$

by using adjointness twice. This verifies the claim we wanted. ▲

The idea is that one can easily map *out* of the value of a left adjoint functor, just as one can map out of a colimit.

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